Solutions in the 2+1 Null Surface Formulation

Tina A. Harriott and J.G. Williams

Abstract The null surface formulation of general relativity (NSF) differs from the standard approach by featuring a function Z, describing families of null surfaces, as the prominent variable, rather than the metric tensor. It is possible to reproduce the metric, to within a conformal factor, by using Z (entering through its third derivative, which is denoted by Λ) and an auxiliary function Ω. The functions Λ and Ω depend upon the spacetime coordinates, which are usually introduced in a manner that is convenient for the null surfaces, and also upon an additional angular variable. A brief summary of the (2+1)-dimensional null surface formulation is presented, together with the NSF field equations for Λ and Ω. A few special solutions are found and the properties of one of them are explored in detail.

1 Introduction

Frittelli, Kozameh and Newman [1, 2, 3] have introduced an alternative approach to general relativity called the null surface formulation (NSF). In this approach, it is not the metric g_{ab} that plays a primary role, but a function Z, which is used to specify families of null surfaces. If needed, a metric can be constructed up to a conformal factor from a knowledge of Z and an auxiliary function Ω. A (2+1)-dimensional version of the NSF has been developed by Forni, Iriondo, Kozameh and Parisi [4, 5], Tanimoto [6] and Silva-Ortigoza [7]. Central to the NSF in 2+1 dimensions is a third-order ordinary differential equation,

$$u''' = \Lambda(u, u', u'', \varphi),$$

Tina A. Harriott and J.G. Williams
Department of Mathematics and Computer Science, Mount Saint Vincent University, Halifax, Nova Scotia B3M 2J6, Canada; Department of Mathematics and Computer Science, and the Winnipeg Institute for Theoretical Physics, Brandon University, Brandon, Manitoba R7A 6A9, Canada
e-mail: tina.harriott@msvu.ca, williams@brandonu.ca

where the prime denotes differentiation with respect to the angular variable $\varphi \in S^1$. Solutions are written $u = Z(x^a;\varphi)$ with x^a $(a = 0,1,2)$ representing three constants of integration which are to be identified with coordinates in (2+1)-dimensional spacetime.

The NSF uses *intrinsic* coordinates [2],

$$\begin{aligned} u \equiv \theta^0 \;&:=\; Z(x^a;\varphi),\\ \omega \equiv \theta^1 \;&:=\; u' \;\equiv\; \partial u \;\equiv\; \partial Z(x^a;\varphi),\\ \rho \equiv \theta^2 \;&:=\; u'' \;\equiv\; \partial^2 u \;\equiv\; \partial^2 Z(x^a;\varphi), \end{aligned}$$

(where $\partial := \partial/\partial\varphi$ denotes the derivative with respect to φ when x^a is held fixed) to derive field equations that are consistent with general relativity,

$$2[\partial(\partial_\rho\Lambda) - \partial_\omega\Lambda - \tfrac{2}{9}(\partial_\rho\Lambda)^2]\partial_\rho\Lambda - \partial^2(\partial_\rho\Lambda) + 3\,\partial(\partial_\omega\Lambda) - 6\,\partial_u\Lambda = 0,$$

$$3\,\partial\Omega \;=\; \Omega\,\partial_\rho\Lambda, \qquad\qquad \partial_\rho^2\Omega \;=\; \kappa T_{\rho\rho}\Omega.$$

2 Nontrivial solution

In the present paper, instead of using our previous light cone cut approach [8], we find a nontrivial solution directly by making the simplifying assumption that Λ and Ω depend only upon ρ: $\Lambda = \Lambda(\rho)$ and $\Omega = \Omega(\rho)$. This implies $\Omega = \Lambda^{1/3}$. For further simplicity, assume that Λ takes the particular form $\Lambda = (a+\rho)^k$ where a and k are constants. This leads to the quadratic, $(2/9)\,k^2 - k + 1 = 0$, which has solutions $k = 3$ and $k = 3/2$. Ignoring the choice $k = 3$ (which leads to empty space), we choose $k = 3/2$. This gives the solution

$$\Lambda = (a+\rho)^{3/2}, \quad \Omega = (a+\rho)^{1/2},$$

with a nonzero source term,

$$T_{\rho\rho} = -\frac{1}{4\kappa\,(a+\rho)^2},$$

and corresponds to the metric

$$ds^2 = (a+\rho)^{-1}\left[\,\tfrac{1}{4}(a+\rho)\,du^2 \;+(a+\rho)^{1/2}\,du d\omega \;-\; 2\,du d\rho + d\omega^2\,\right].$$

The three independent curvature scalars of 2+1 dimensions are found to be

$$R \;=\; \frac{1}{32}, \qquad R_{ab}R^{ab} \;=\; \frac{3}{1024}, \qquad \frac{\det\|R_{ab}\|}{\det\|g_{ab}\|} \;=\; -\left(\frac{1}{32}\right)^3,$$

and the components of the Einstein tensor are

$$G_{uu} = -\frac{3}{256}, \qquad G_{u\omega} = -\frac{3}{128}(a+\rho)^{-1/2}, \qquad G_{u\rho} = \frac{3}{64}(a+\rho)^{-1},$$
$$G_{\omega\omega} = -\frac{3}{64}(a+\rho)^{-1}, \quad G_{\omega\rho} = \frac{1}{8}(a+\rho)^{-3/2}, \quad G_{\rho\rho} = -\frac{1}{4}(a+\rho)^{-2}.$$

The null surface formulation of general relativity does not distinguish between conformally related spacetimes, and so a conformally flat spacetime would be an uninteresting example. The Cotton-York tensor C_{ab} is nonzero for the above solution, indicating that the spacetime is not conformally flat:

$$C_{uu} = -\frac{1}{256}, \qquad C_{u\omega} = -\frac{1}{128}(a+\rho)^{-1/2}, \qquad C_{u\rho} = \frac{1}{64}(a+\rho)^{-1},$$

$$C_{\omega\omega} = -\frac{1}{64}(a+\rho)^{-1}, \quad C_{\omega\rho} = \frac{3}{64}(a+\rho)^{-3/2}, \quad C_{\rho\rho} = -\frac{3}{32}(a+\rho)^{-2}.$$

In 2+1 dimensions, the Einstein equations, $G_{ab} = \kappa T_{ab}$, are sometimes replaced by the Einstein-Cotton field equations of topologically massive gravity (thereby allowing gravitational excitations):

$$G_{ab} + \lambda g_{ab} + \frac{1}{m} C_{ab} = \kappa T_{ab}.$$

The constant m can take either sign. (In fact, in 2+1 dimensions, this is also true for κ). It is straightforward to show that the metric under consideration satisfies the field equations of topologically massive gravity for a perfect fluid source, $T_{ab} = (\mu + p)U_a U_b + p\, g_{ab}$, with velocity U_a given by

$$U_u = 0, \qquad U_\omega = (a+\rho)^{-1/2}, \qquad U_\rho = -2(a+\rho)^{-1},$$

and with constant μ and p. Specifically:

$$m = -3/8, \quad \mu = -p, \quad p = \frac{1}{\kappa}\left(\lambda - \frac{1}{192}\right).$$

The most interesting case comes from choosing $\lambda = 1/192$. This gives a topologically massive gravity solution analogous to the regular de Sitter solution: a vacuum solution with nonzero cosmological constant and nonzero expansion θ.

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