Solutions in the 2+1 Null Surface Formulation

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Abstract The null surface formulation of general relativity (NSF) differs from the standard approach by featuring a function $Z$, describing families of null surfaces, as the prominent variable, rather than the metric tensor. It is possible to reproduce the metric, to within a conformal factor, by using $Z$ (entering through its third derivative, which is denoted by $A$) and an auxiliary function $\Omega$. The functions $A$ and $\Omega$ depend upon the spacetime coordinates, which are usually introduced in a manner that is convenient for the null surfaces, and also upon an additional angular variable. A brief summary of the (2+1)-dimensional null surface formulation is presented, together with the NSF field equations for $A$ and $\Omega$. A few special solutions are found and the properties of one of them are explored in detail.

1 Introduction

Frittelli, Kozameh and Newman [1, 2, 3] have introduced an alternative approach to general relativity called the null surface formulation (NSF). In this approach, it is not the metric $g_{ab}$ that plays a primary role, but a function $Z$, which is used to specify families of null surfaces. If needed, a metric can be constructed up to a conformal factor from a knowledge of $Z$ and an auxiliary function $\Omega$. A (2+1)-dimensional version of the NSF has been developed by Forni, Iriondo, Kozameh and Parisi [4, 5], Tanimoto [6] and Silva-Ortigoza [7]. Central to the NSF in 2+1 dimensions is a third-order ordinary differential equation,

$$u''' = A(u, u', u'', \varphi),$$
where the prime denotes differentiation with respect to the angular variable \( \phi \in S^1 \).

Solutions are written \( u = Z(x^a; \phi) \) with \( x^a (a = 0, 1, 2) \) representing three constants of integration which are to be identified with coordinates in (2+1)-dimensional spacetime.

The NSF uses intrinsic coordinates \([2]\),

\[
\begin{align*}
  u &\equiv \theta^0 := Z(x^a; \phi), \\
  \omega &\equiv \theta^1 := u' \equiv \partial u \equiv \partial Z(x^a; \phi), \\
  \rho &\equiv \theta^2 := u'' \equiv \partial^2 u \equiv \partial^2 Z(x^a; \phi),
\end{align*}
\]

(where \( \partial := \partial / \partial \phi \) denotes the derivative with respect to \( \phi \) when \( x^a \) is held fixed) to derive field equations that are consistent with general relativity,

\[
2[\partial (\partial_{\rho} \Lambda) - \partial_{\rho} \Lambda - \frac{7}{3} (\partial_{\rho} \Lambda)^2] \partial_{\rho} \Lambda - \partial^2 (\partial_{\rho} \Lambda) + 3 \partial (\partial_{\rho} \Lambda) - 6 \partial_{\rho} \Lambda = 0,
\]

\[
3 \partial \Omega = \Omega \partial_{\rho} \Lambda, \quad \partial_{\rho}^2 \Omega = \kappa T_{\rho\rho} \Omega.
\]

### 2 Nontrivial solution

In the present paper, instead of using our previous light cone cut approach \([8]\), we find a nontrivial solution directly by making the simplifying assumption that \( \Lambda \) and \( \Omega \) depend only upon \( \rho \): \( \Lambda = \Lambda(\rho) \) and \( \Omega = \Omega(\rho) \). This implies \( \Omega = \Lambda^{1/3} \). For further simplicity, assume that \( \Lambda \) takes the particular form \( \Lambda = (a + \rho)^k \) where \( a \) and \( k \) are constants. This leads to the quadratic, \((2/9) k^2 - k + 1 = 0\), which has solutions \( k = 3 \) and \( k = 3/2 \). Ignoring the choice \( k = 3 \) (which leads to empty space), we choose \( k = 3/2 \). This gives the solution

\[
\Lambda = (a + \rho)^{3/2}, \quad \Omega = (a + \rho)^{1/2},
\]

with a nonzero source term,

\[
T_{\rho\rho} = -\frac{1}{4\kappa (a + \rho)^2},
\]

and corresponds to the metric

\[
d s^2 = (a + \rho)^{-1} \left[ \frac{1}{4} (a + \rho) \; d u^2 + (a + \rho)^{1/2} \; d u d \omega - 2 \; d u d \rho + d \omega^2 \right] .
\]

The three independent curvature scalars of 2+1 dimensions are found to be

\[
R = \frac{1}{32}, \quad R_{ab} R^{ab} = \frac{3}{1024}, \quad \frac{\det \| R_{ab} \|}{\det \| g_{ab} \|} = -\left( \frac{1}{32} \right)^3 ,
\]

and the components of the Einstein tensor are
The null surface formulation of general relativity does not distinguish between conformally related spacetimes, and so a conformally flat spacetime would be an uninteresting example. The Cotton-York tensor \( C_{ab} \) is nonzero for the above solution, indicating that the spacetime is not conformally flat:

\[
G_{uu} = -\frac{3}{256}, \quad G_{uw} = -\frac{3}{128}(a + \rho)^{-1/2}, \quad G_{up} = \frac{3}{64}(a + \rho)^{-1}, \\
G_{wv} = -\frac{3}{64}(a + \rho)^{-1}, \quad G_{wp} = \frac{3}{8}(a + \rho)^{-3/2}, \quad G_{pp} = -\frac{1}{4}(a + \rho)^{-2}.
\]

In 2+1 dimensions, the Einstein equations, \( G_{ab} = \kappa T_{ab} \), are sometimes replaced by the Einstein-Cotton field equations of topologically massive gravity (thereby allowing gravitational excitations):

\[
G_{ab} + \lambda g_{ab} + \frac{1}{m}C_{ab} = \kappa T_{ab}.
\]

The constant \( m \) can take either sign. (In fact, in 2+1 dimensions, this is also true for \( \kappa \).) It is straightforward to show that the metric under consideration satisfies the field equations of topologically massive gravity for a perfect fluid source, \( T_{ab} = (\mu + p)U_a U_b + p g_{ab} \), with velocity \( U_a \) given by

\[
U_u = 0, \quad U_w = (a + \rho)^{-1/2}, \quad U_p = -2(a + \rho)^{-1},
\]

and with constant \( \mu \) and \( p \). Specifically:

\[
m = -3/8, \quad \mu = -p, \quad p = \frac{1}{\kappa} \left( \lambda - \frac{1}{192} \right).
\]

The most interesting case comes from choosing \( \lambda = 1/192 \). This gives a topologically massive gravity solution analogous to the regular de Sitter solution: a vacuum solution with nonzero cosmological constant and nonzero expansion \( \theta \).

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References