Dynamic and Thermodynamic Stability of Black Holes and Black Branes

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Abstract I describe recent work with with Stefan Hollands that establishes a new criterion for the dynamical stability of black holes in $D \geq 4$ spacetime dimensions in general relativity with respect to axisymmetric perturbations: Dynamical stability is equivalent to the positivity of the canonical energy, $\mathscr{E}$, on a subspace of linearized solutions that have vanishing linearized ADM mass, momentum, and angular momentum at infinity and satisfy certain gauge conditions at the horizon. We further show that $\mathscr{E}$ is related to the second order variations of mass, angular momentum, and horizon area by $\mathscr{E} = \delta^2 M - \sum_i \Omega_i \delta^2 J_i - (\kappa/8\pi)\delta^2 A$, thereby establishing a close connection between dynamical stability and thermodynamic stability. Thermodynamic instability of a family of black holes need not imply dynamical instability because the perturbations towards other members of the family will not, in general, have vanishing linearized ADM mass and/or angular momentum. However, we prove that all black branes corresponding to thermodynmically unstable black holes are dynamically unstable, as conjectured by Gubser and Mitra. We also prove that positivity of $\mathscr{E}$ is equivalent to the satisfaction of a "local Penrose inequality," thus showing that satisfaction of this local Penrose inequality is necessary and sufficient for dynamical stability.

It is of considerable interest to determine the linear stablity of black holes in (D-dimensional) general relativity. It is also of interest to determine the linear stability of the corresponding black branes in $(D+p)$-dimensions, i.e., spacetimes with metric of the form

$$d\tilde{s}^2_{D+p} = ds^2_D + \sum_{i=1}^{p} dz_i^2 \,, \tag{1}$$

where ds^2_D is a black hole metric. In this paper, I will describe some recent general results, obtained in collaboration with Stefan Hollands, on the stability of black holes and black branes. In our work, we restrict consideration to vacuum general

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relativity without a cosmological constant, but our methods are applicable general theories of gravity derived from a diffeomorphism covariant Lagrangian. A full account of our results can be found in [1].

One can analyze the stability of a black hole or black brane by writing out the linearized Einstein equation off of the black hole or black brane background spacetime. One can establish linear stability by finding a positive definite conserved norm for perturbations. Linear instability can be established by finding a solution with (gauge independent) unbounded growth in time. However, even in the very simplest cases—such as the Schwarzschild black hole [2, 3] and the Schwarzschild black string [4]—it is quite nontrivial to carry out the decoupling of equations and the fixing of gauge needed to determine stability or instability directly from the equations of motion. Furthermore, since this analysis depends on the details of the equations of motion, it must be done on a case-by-case basis. Thus, it would be useful to have a much simpler criterion for stability that can be applied to any black hole or black brane.

In ordinary thermodynamics, one has a very simple and general criterion for thermodynamic instability of a homogeneous system in thermal equilibrium. Consider such a system, whose entropy, S, is a function of energy, E, and other extensive state parameters X_i, so that $S = S(E, X_i)$. The condition for thermodynamic instability is that the Hessian matrix

$$\mathbf{H}_S = \begin{pmatrix} \frac{\partial^2 S}{\partial E^2} & \frac{\partial^2 S}{\partial X_i \partial E} \\ \frac{\partial^2 S}{\partial E \partial X_i} & \frac{\partial^2 S}{\partial X_i \partial X_j} \end{pmatrix} . \tag{2}$$

admit a positive eigenvalue[1]. This criterion arises from the fact that if the Hessian had a positive eigenvalue, then one could increase total entropy by exchanging E and/or X_i between different parts of the system. To see this more explicitly, let $\xi_0 = (E_0, X_{i0})$ denote the parameter values of a particular thermodynamic state, let $\mathbf{v}$ be an arbitrary vector in the thermodynamic state space, and consider the one-parameter family $\xi(\lambda) = \xi_0 + \lambda \mathbf{v}$ of thermodynamic states. It is obvious that for this family, we have $d^2E/d\lambda^2 = d^2X_i/d\lambda^2 = 0$, whereas $d^2S/d\lambda^2|_0 = \mathrm{Hess}_S|_{\xi_0}(\mathbf{v}, \mathbf{v})$. Suppose now that, for our homogeneous system, we change the state parameters by $\lambda \mathbf{v}$ in one part of the system and compensate for this by changing the state parameters by $-\lambda \mathbf{v}$ in a different part of the system (of the same "size"). To first order in λ, there will be no change in the total entropy. To second order in λ, the change in total entropy will be proportional to $\mathrm{Hess}_S|_{\xi_0}(\mathbf{v}, \mathbf{v})$. Thus, if $\mathbf{H}_S$ admits a positive eigenvalue, one can choose $\mathbf{v}$ so as to find a state of higher entropy at fixed (E, X_i) arbitrarily close to the original thermal equilibrium state.

An equivalent statement of the criterion for thermodynamic instability of the state $\xi_0 = (E_0, X_{i0})$ is (assuming $T > 0$) that one can find a one parameter family $\xi(\lambda)$ of thermal equilibrium states with $\xi(0) = \xi_0$ such that

$$\delta^2 E - T\delta^2 S - \sum_i Y_i \delta^2 X_i < 0 \tag{3}$$

[1] Note that for the case where E is the only state parameter, this criterion is equivalent to the system having a negative heat capacity

where $Y_i \equiv (\partial E/\partial X_i)_S$ and δ^2 denotes $d^2/d\lambda^2$ evaluated at $\lambda = 0$. To see this, we note that for a one parameter family of the form $\xi_0 + \lambda \mathbf{v}$, the left side is just $-T\mathrm{Hess}_S|_{\xi_0}(\mathbf{v},\mathbf{v})$. However, by the first law of thermodynamics (i.e., the definitions of Y_i and $T^{-1} \equiv (\partial S/\partial E)_{X_i}$), the left side does not depend on the second order change in the state, so eq.(3) holds for the family $\xi(\lambda)$ if and only if it holds for the family $\xi_0 + \lambda \mathbf{v}$ with $\mathbf{v} = d\xi/d\lambda|_0$.

Black holes are thermodynamic systems, with

$$\begin{aligned} E &\leftrightarrow M \\ S &\leftrightarrow \frac{A}{4} \\ X_i &\leftrightarrow J_i, Q_i \end{aligned} \tag{4}$$

Thus, in the vacuum case ($Q_i = 0$) a black hole would be said to be thermodynamically unstable if the Hessian matrix

$$\mathbf{H}_A = \begin{pmatrix} \frac{\partial^2 A}{\partial M^2} & \frac{\partial^2 A}{\partial J_i \partial M} \\ \frac{\partial^2 A}{\partial M \partial J_i} & \frac{\partial^2 A}{\partial J_i \partial J_j} \end{pmatrix}. \tag{5}$$

admits a positive eigenvalue. This is equivalent to finding a perturbation for which

$$\delta^2 M - \frac{\kappa}{8\pi}\delta^2 A - \sum_i \Omega_i \delta^2 J_i < 0\,. \tag{6}$$

One might expect that this condition for thermodynamic instability might imply dynamical instability. However, this is clearly false: The Schwarzschild black hole has negative heat capacity ($A = 16\pi M^2$, so $\partial^2 A/\partial M^2 = 32\pi > 0$) but is well known to be dynamically stable. A black hole is not “homogeneous” in a manner that would allow one to borrow energy and/or angular momentum from one part of it and give it to another part in such a way as to increase the total entropy (area) at fixed total energy and angular momentum in the manner described above for thermodynamic systems.

However, a black brane is potentially homogeneous in this sense, and the Schwarzschild black string is known to be unstable [4]. The Gubser-Mitra conjecture [5], [6] states that the above thermodynamic criterion for instability is a valid criterion dynamical instability for black branes. As described further below, our work provides a proof of the Gubser-Mitra conjecture, which follows as a consequence of a more fundamental stability criterion that we shall establish.

Another simple possible stability criterion that is applicable to black holes is the “local Penrose inequality,” discussed in [7]. We reformulate this criterion as follows: Suppose one has a family of stationary, axisymmetric black holes parametrized by M and angular momenta $J_1, \ldots, J_N$. Consider a one-parameter family $g_{ab}(\lambda)$ of axisymmetric spacetimes, with $g_{ab}(0)$ being a member of this family with surface gravity $\kappa > 0$. Consider initial data on a hypersurface Σ passing through the bifurcation surface B. By the linearized Raychauduri equation, to first order in λ, the event horizon coincides with the apparent horizon on Σ. They need not coincide to second order in λ, but since B is an extremal surface in the background spacetime, their areas must agree to second order. Let $\mathscr{A}(\lambda)$ denotes the area of the apparent horizon of $g_{ab}(\lambda)$, and let $\bar{A}(\lambda)$ denote the area of the event horizon of the stationary black hole in the family with the same mass and angular momentum as $g_{ab}(\lambda)$. Suppose that to second order, we have

$$\delta^2 \mathscr{A} > \delta^2 \bar{A}$$

Since (i) the area of the event horizon can only increase with time (by cosmic censorship), (ii) the final mass of the black hole cannot be larger than the initial total mass (by positivity of Bondi flux), (iii) its final angular momenta must equal the initial angular momenta (by axisymmetry), and (iv) $\bar{A}(M, J_1, \ldots, J_N)$ is an increasing function of M at fixed J_i (by the first law of black hole mechanics with $\kappa > 0$), it follows that there would be a contradiction if the perturbed black hole solution were to settle down to a stationary black hole in the family. This implies that satisfaction of this inequality implies instablity—although it does not imply stability if $\delta^2 \mathscr{A} \leq \delta^2 \bar{A}$ always holds. As discussed further below, our more fundamental stability criterion implies that satisfaction of $\delta^2 \mathscr{A} \leq \delta^2 \bar{A}$ is necessary and sufficient for the dynamical stability of black holes with respect to axisymmetric perturbations.

Our results are based upon identities arising from the Lagrangian formulation of general relativity. Although we restrict consideration here to vacuum general relativity, these formulas can be generalized to allow for the presence of matter fields and, indeed, they can be generalized to an arbitrary diffeomorphism covariant theory of gravity [8], provided only that the field equations are derived from a Lagrangian. The key identities we use are obtained as follows:

The Lagrangian D-form for vacuum general relativity in D dimensions is

$$L_{a_1 \ldots a_D} = \frac{1}{16\pi} R \, \varepsilon_{a_1 \ldots a_D} . \tag{7}$$

Its first variation yields

$$\delta L = E \cdot \delta g + d\theta , \tag{8}$$

where $E = 0$ is the vacuum Einstein field equation and the $(D-1)$-form $\theta(g, \delta g)$ is the "boundary term" that is usually discarded when the variation of L is performed under an integral sign. Explicitly, we have

$$\theta_{a_1 \ldots a_{D-1}} = \frac{1}{16\pi} g^{ac} g^{bd} (\nabla_d \delta g_{bc} - \nabla_c \delta g_{bd}) \varepsilon_{a a_1 \ldots a_{D-1}} . \tag{9}$$

The symplectic current $(D-1)$-form is defined by

$$\omega(g; \delta_1 g, \delta_2 g) = \delta_1 \theta(g; \delta_2 g) - \delta_2 \theta(g; \delta_1 g) . \tag{10}$$

The symplectic form, $W_\Sigma(g; \delta_1 g, \delta_2 g)$, is obtained by integrating ω over a Cauchy surface Σ

$$W_\Sigma(g; \delta_1 g, \delta_2 g) \equiv \int_\Sigma \omega(g; \delta_1 g, \delta_2 g) . \tag{11}$$

It can be shown to be given by [9]

$$W_\Sigma(g;\delta_1 g,\delta_2 g) = -\frac{1}{32\pi}\int_\Sigma (\delta_1 h_{ab}\delta_2 p^{ab} - \delta_2 h_{ab}\delta_1 p^{ab}), \tag{12}$$

where

$$p^{ab} \equiv h^{1/2}(K^{ab} - h^{ab}K), \tag{13}$$

where K_{ab} is the extrinsic curvature of Σ.

The Lagrangian L, eq.(7), is diffeomorphism covariant. An arbitrary vector field X^a is the generator of an infinitesimal diffeomorphism, and associated to X^a is a conserved Noether current $(D-1)$-form, defined by

$$\mathscr{J}_X \equiv \theta(g,\mathscr{L}_X g) - X\cdot L, \tag{14}$$

where $X\cdot L$ denotes the $(D-1)$-form $X^a L_{aa_1\ldots a_{D-1}}$. It can be shown quite generally [10] that $\mathscr{J}_X$ always can be written in the form

$$\mathscr{J}_X = X\cdot C + dQ_X, \tag{15}$$

where $C=0$ are the constraint equations [11] of the theory, and where the $(D-2)$-form Q_X is called the *Noether charge*.

We now take the first variation of $\mathscr{J}_X$, using eqs.(14) and (15) as well as eqs.(8) and (10). We thereby obtain the following fundamental variational identity:

$$\omega(g;\delta g,\mathscr{L}_X g) = X\cdot[E(g)\cdot\delta g] + X\cdot\delta C + d\,[\delta Q_X(g) - X\cdot\theta(g;\delta g)] \tag{16}$$

It should be emphasized that eq.(16) holds for an arbitrary metric g_{ab} (not necessarily a solution to the field equations), an arbitrary metric perturbation δg_{ab} (not necessarily a solution to the linearized field equations) and an arbitrary vector field X^a.

By definition, a Hamiltonian, h_X, for the "time evolution" generated by X^a is a function on phase space whose first variation satisfies

$$\delta h_X = W_\Sigma(g;\delta g,\mathscr{L}_X g) \tag{17}$$

if and only if g_{ab} satisfies the equations of motion $E=0$. By eq.(16), if a Hamiltonian h_X conjugate to X^a exists, its first variation must satisfy

$$\delta h_X = \int_\Sigma (X\cdot\delta C + d\,[\delta Q_X(g) - X\cdot\theta(g;\delta g)]) \tag{18}$$

For asymptotically flat spacetimes, this motivates the definition of the ADM conserved quantity, H_X, associated with an asymptotic symmetry X^a, as the quantitiy defined for solutions whose first variation is given by

$$\delta H_X = \int_\infty [\delta Q_X(g) - X\cdot\theta(g;\delta g)] \tag{19}$$

Now consider a stationary black hole solution ($E = 0$) with surface gravity $\kappa > 0$, so the event horizon is of "bifurcate type," with bifurcation surface B. Let Σ be a Cauchy surface for the exterior region, so that it extends from spatial infinity to B. We choose X to be the horizon Killing field

$$K^a = t^a + \sum \Omega_i \phi_i^a \,. \tag{20}$$

Finally, let $\gamma = \delta g$ satisfy the linearized constraint equations $\delta C = 0$. Integration of the fundamental identity (16) over Σ—using $\mathscr{L}_X g = 0$, $E = 0$, and $\delta C = 0$—then yields the first law of black hole mechanics [8]

$$0 = \delta M - \sum_i \Omega_i \delta J_i - \frac{\kappa}{8\pi} \delta A \,. \tag{21}$$

To proceed further, we impose two gauge conditions at B on our perturbation $\gamma = \delta g$. The first condition

$$\delta \vartheta|_B = 0 \tag{22}$$

ensures that the location of the horizon does not change to first order. (Here ϑ_B denotes the expansion of the outgoing null geodesics from B.) The second condition is

$$\delta \varepsilon|_B = \frac{\delta A}{A} \varepsilon \,, \tag{23}$$

where $\varepsilon|_B$ denotes the surface area element on B. The imposition of these conditions does not involve any loss of generality, i.e., they can be imposed for arbitrary perturbations [1].

We define the *canonical energy* of a perturbation γ by

$$\mathscr{E} \equiv W_\Sigma \left(g; \gamma, \mathscr{L}_t \gamma\right) \,. \tag{24}$$

The second variation of our fundamental identity (16) then yields (for axisymmetric perturbations)

$$\mathscr{E} = \delta^2 M - \sum_i \Omega_i \delta^2 J_i - \frac{\kappa}{8\pi} \delta^2 A \,. \tag{25}$$

Thus positivity of $\mathscr{E}$ for all perturbations γ is equivalent to thermodynamic stability (see eq.(6)).

Our results on dynamical stability follow from various properties of $\mathscr{E}$. To establish these properties, it is useful to view $\mathscr{E}$ as a quadratic form on perturbations

$$\mathscr{E}(\gamma_1, \gamma_2) = W_\Sigma(g; \gamma_1, \mathscr{L}_t \gamma_2) \tag{26}$$

In [1], we proved that $\mathscr{E}$ satisfies the following properties:

- $\mathscr{E}$ is conserved, i.e., it takes the same value if evaluated on another Cauchy surface Σ' extending from spatial infinity to B.
- $\mathscr{E}$ is symmetric, $\mathscr{E}(\gamma_1, \gamma_2) = \mathscr{E}(\gamma_2, \gamma_1)$.

- When restricted to perturbations for which $\delta A = 0$ and $\delta P_i = 0$ (where P_i is the ADM linear momentum), $\mathscr{E}$ is gauge invariant.
- When restricted to the subspace, $\mathscr{V}$, of perturbations for which $\delta M = \delta J_i = \delta P_i = 0$ (and hence, by the first law of black hole mechanics $\delta A = 0$), we have $\mathscr{E}(\gamma', \gamma) = 0$ for all $\gamma' \in \mathscr{V}$ if and only if γ is a perturbation towards another stationary and axisymmetric black hole.

Thus, if we restrict to perturbations in the subspace, $\mathscr{V}'$, of perturbations in $\mathscr{V}$ modulo perturbations towards other stationary black holes, then $\mathscr{E}$ is a non-degenerate quadratic form. Consequently, on $\mathscr{V}'$, either (a) $\mathscr{E}$ is positive definite or (b) there is a $\psi \in \mathscr{V}'$ such that $\mathscr{E}(\psi) < 0$. If (a) holds, then $\mathscr{E}$ provides a positive definite conserved norm on perturbations. *Thus, if (a) holds, we have stability.*

To analyze case (b), we must consider the flux of $\mathscr{E}$ through null infinity, $\mathscr{I}^+$, and through the black hole horizon, $\mathscr{H}$. Let δN_{ab} denote the perturbed Bondi news tensor at null infinity and let $\delta\sigma_{ab}$ denote the perturbed shear on the horizon. If the perturbed black hole were to "settle down" to another stationary black hole at late times, then $\delta N_{ab} \to 0$ and $\delta\sigma_{ab} \to 0$ at late times. In [1], we showed that—for axisymmetric perturbations—the change in canonical energy is then given by

$$\Delta\mathscr{E} = -\frac{1}{16\pi}\int_{\mathscr{I}} \delta\tilde{N}_{cd}\delta\tilde{N}^{cd} - \frac{1}{4\pi}\int_{\mathscr{H}} (K^a\nabla_a u)\, \delta\sigma_{cd}\delta\sigma^{cd} \;\leq 0\,. \tag{27}$$

Thus, $\mathscr{E}$ can only decrease. Therefore if one has a perturbation $\psi \in \mathscr{V}$ such that $\mathscr{E}(\psi) < 0$, then ψ cannot "settle down" to a stationary solution at late times because $\mathscr{E} = 0$ for stationary perturbations with $\delta M = \delta J_i = \delta P_i = 0$. *Thus, in case (b) we have instability.*

The above results show that the necessary and sufficient condition for stability of a black hole (or black brane) with respect to axisymmetric perturbations is positivity of $\mathscr{E}$ on a Hilbert space, $\mathscr{V}$, of perturbations with vanishing perturbed mass, angular momentum, and linear momentum, $\delta M = \delta J_i = \delta P_i = 0$. This is our fundamental criterion for the dynamical stability of black holes and black branes. In view of eqs.(25) and (6), it follows that *dynamical stability is equivalent to thermodynamic stability on the subspace of perturbations that satisfy* $\delta M = \delta J_i = \delta P_i = 0$.

The restriction that $\delta M = \delta J_i = \delta P_i = 0$ can be removed for the case of black branes as a consequence of the following theorem [1]:

Theorem 1. *Suppose a family of black holes parametrized by* (M, J_i) *is thermodynamically unstable at* (M_0, J_{0i})*, i.e., there exists a perturbation within the black hole family for which* $\mathscr{E} < 0$. *Then, for any black brane corresponding to* (M_0, J_{0i}) *one can find a sufficiently long wavelength perturbation for which* $\tilde{\mathscr{E}} < 0$ *and* $\delta\tilde{M} = \delta\tilde{J}_i = \delta\tilde{P}_i = \delta\tilde{A} = \delta\tilde{T}_i = 0$ *(where* $\tilde{T}_i$ *denotes the momenta conjugate to the translational symmetries of the brane).*

This theorem is proven by starting with the initial data for the perturbation to another black hole with $\mathscr{E} < 0$, multiplying it by $\exp(ikz)$—where "z" denotes a brane coordinate as in eq.(1)—and then re-adjusting the initial data so that it satisfies the constraints. The new data will automatically satisfy $\delta\tilde{M} = \delta\tilde{J}_A = \delta\tilde{P}_i = \delta\tilde{A} =$

$\delta\tilde{T}_i = 0$ because of the $\exp(ikz)$ factor. For sufficiently small k, it can be shown to satisfy $\tilde{\mathscr{E}} < 0$.

The above theorem, together with our fundamental criterion for dynamical stability, proves the Gubser-Mitra conjecture. To illustrate the nature of this result, consider the one-parameter family of Schwarzschild black holes, parametrized by mass M. It is easily seen that the "change of mass" perturbation has $\mathscr{E} < 0$. However, this tells one nothing about the stability of Schwarzschild black holes because, obviously, for these perturbations we have $\delta M \neq 0$, so they do not "count" for testing stability of the Schwarzschild black hole. However, the fact that $\mathscr{E} < 0$ for this "change of mass" perturbation proves the instability of Schwarzschild black branes to sufficiently long wavelength perturbations.

The equivalence of the satisfaction of the local Penrose inequality to our fundamental stability criterion for black holes can be seen as follows. As above, let $\bar{g}_{ab}(M,J_i)$ be a family of stationary, axisymmetric, and asymptotically flat black hole metrics on M. Let $g_{ab}(\lambda)$ be a one-parameter family of axisymmetric metrics such that $g_{ab}(0) = \bar{g}_{ab}(M_0,J_{0i})$. Let $M(\lambda), J_i(\lambda)$ denote the mass and angular momenta of $g_{ab}(\lambda)$ and let $\mathscr{A}(\lambda)$ denote the area of its apparent horizon. Let $\bar{g}_{ab}(\lambda) = \bar{g}_{ab}(M(\lambda),J_i(\lambda))$ denote the one-parameter family of stationary black holes with the same mass and angular momenta as $g_{ab}(\lambda)$. We have the following result:

Theorem 2. *There exists a one-parameter family $g_{ab}(\lambda)$ for which*

$$\mathscr{A}(\lambda) > \bar{\mathscr{A}}(\lambda) \tag{28}$$

to second order in λ if and only if there exists a perturbation γ'_{ab} of $\bar{g}_{ab}(M_0,(J_{0i})$ with $\delta M = \delta J_i = \delta P_i = 0$ such that $\mathscr{E}(\gamma') < 0$.

Proof. The first law of black hole mechanics implies $\mathscr{A}(\lambda) = \bar{\mathscr{A}}(\lambda)$ to first order in λ, so what counts are the second order variations. Since the families have the same mass and angular momenta, we have

$$\begin{aligned} \frac{\kappa}{8\pi}\left[\frac{d^2\mathscr{A}}{d\lambda^2}(0) - \frac{d^2\bar{\mathscr{A}}}{d\lambda^2}(0)\right] &= \mathscr{E}(\bar{\gamma},\bar{\gamma}) - \mathscr{E}(\gamma,\gamma) \\ &= -\mathscr{E}(\gamma',\gamma') + 2\mathscr{E}(\gamma',\bar{\gamma}) \\ &= -\mathscr{E}(\gamma',\gamma') \end{aligned}$$

where $\gamma' = \bar{\gamma} - \gamma$. □

In summary, the remarkable relationship between the laws of black hole physics and the laws of thermodynamics has been shown to extend to dynamical stability.

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