A Class of Conformal Curves on Spherically Symmetric Spacetimes

Juan A. Valiente Kroon and Christian Lübbe

Abstract A class of curves with special conformal properties (conformal curves) is studied on the Reissner-Nordström spacetime. It is shown that initial data for the conformal curves can be prescribed so that the resulting congruence of curves does not contain any conjugate points in the domain of outer communication of the spacetime and extend smoothly to future and past null infinity. The results of this analysis are expected to be of relevance for the discussion of the Reissner-Nordström and other spherically symmetric spacetimes as solutions to the conformal field equations and for the numerical global evaluation of static black hole spacetimes.

1 Introduction

Conformal methods constitute a powerful tool for the discussion of global properties of spacetimes —in particular, those representing black holes. The conformal structure of static electrovacuum black hole spacetimes is, to some extent, well understood —see e.g. [1, 2]. However, the constructions involved often require several changes of variables and the introduction of some type of null coordinates. This choice of coordinates may not be the most convenient to undertake an analysis of global or asymptotic properties of a spacetime by means of the *conformal Einstein field equations* —see e.g. the discussion in [3]. A key issue in this respect, is how to construct in a systematic or canonical fashion a conformal extension of the spacetime which, in addition, eases the analysis of the underlying conformal

Juan A. Valiente Kroon
School of Mathematical Sciences, Queen Mary University of London
Mile End Road, London E1 4NS, UK
e-mail: j.a.valiente-kroon@qmul.ac.uk

Christian Lübbe
Department of Mathematics, University College London
Gower Street, London, WC1E 6BT, UK

field equations —for a review of the conformal equations and the issues involved in their analysis see e.g. [4]. In the case of vacuum spacetimes, gauges based on the use of *conformal geodesics* offer such a systematic approach —see e.g. [5, 6]. Conformal geodesics are invariants of the conformal structure: a conformal transformations map conformal geodesics into a conformal geodesics —this is not the case with standard geodesics unless they are null.

One of the main advantages of the use of conformal geodesics in the construction of gauge (and coordinate) systems in a vacuum spacetime is that they provide an *a priori* conformal factor which can be read off directly from the data one has specified to generate the congruence of conformal geodesics. Hence, one has a canonical procedure to generate a conformal extension of the spacetime in question. In addition, gauge systems based on conformal geodesics give rise to a fairly straightforward hyperbolic reduction of the conformal Einstein field equations in which most of the evolution equations are, in fact, transport equations —see e.g. [4, 7].

The useful property of having an *a priori* conformal factor is lost when one considers conformal geodesics in non-vacuum spacetimes. Nevertheless, in [8] it has been shown that this property can be recovered if one considers a more general class of curves —the *conformal curves*. These conformal curves satisfy equations similar to the conformal geodesic equations, but with a different coupling to the curvature of the spacetime. In the vacuum case they coincide with the conformal geodesic equations. Gauges based on this class of curves have been used in [8] to revisit the stability proofs for the Minkowski and the de Sitter spacetimes given in [9] and to obtain a stability result for purely radiative electrovacuum spacetimes. An important remark concerning these results is that they deal with spacetimes, or are restricted to regions of spacetimes, that do not contain singularities.

2 Conformal curves

Given an interval $I \subseteq \mathbb{R}$, let $\boldsymbol{x}(\tau)$, with $\tau \in I$, denote a curve in a spacetime $(\mathscr{M}, \tilde{\boldsymbol{g}})$ and let $\boldsymbol{b}(\tau)$ denote a 1-form along $\boldsymbol{x}(\tau)$. Furthermore, let $\dot{\boldsymbol{x}} \equiv \mathrm{d}\boldsymbol{x}/\mathrm{d}\tau$ denote the tangent vector field of the curve $\boldsymbol{x}(\tau)$. The pairing between the vector $\dot{\boldsymbol{x}}$ and the 1-form $\boldsymbol{b}$ is denoted by $\langle \boldsymbol{b}, \dot{\boldsymbol{x}} \rangle$. In [8] the following equations have been introduced for the pair $(\boldsymbol{x}(\tau), \boldsymbol{b}(\tau))$:

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}} \dot{\boldsymbol{x}} = -2\langle \boldsymbol{b}, \dot{\boldsymbol{x}} \rangle \dot{\boldsymbol{x}} + \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) \boldsymbol{b}^{\sharp}, \tag{1a}$$

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}} \boldsymbol{b} = \langle \boldsymbol{b}, \dot{\boldsymbol{x}} \rangle \boldsymbol{b} - \tfrac{1}{2} \tilde{\boldsymbol{g}}^{\sharp}(\boldsymbol{b}, \boldsymbol{b}) \dot{\boldsymbol{x}}^{\flat} + \tilde{\boldsymbol{H}}(\dot{\boldsymbol{x}}, \cdot), \tag{1b}$$

where $\tilde{\nabla}_{\dot{\boldsymbol{x}}}$ denotes the directional derivative of the Levi-Civita connection of the metric $\tilde{\boldsymbol{g}}$, while $\tilde{\boldsymbol{H}}$ denotes a rank 2 covariant tensor which upon the conformal transformation $\boldsymbol{g} = \Theta^2 \tilde{\boldsymbol{g}}$ transforms as:

$$\tilde{H}_{\mu\nu} - H_{\mu\nu} = \nabla_\mu \Upsilon_\nu + \Upsilon_\mu \Upsilon_\nu - \tfrac{1}{2} g^{\lambda\rho} \Upsilon_\lambda \Upsilon_\rho \, g_{\mu\nu},$$

where $\Upsilon_\mu \equiv \Theta^{-1}\nabla_\mu \Theta$. This transformation law is formally identical to that of the *Schouten tensor*

$$\tilde{L}_{\mu\nu} \equiv \tfrac{1}{2}(\tilde{R}_{\mu\nu} - \tfrac{1}{6}\tilde{R}\tilde{g}_{\mu\nu}).$$

The equations (1a)-(1b) are called the *conformal curve equations*. In [8] the choice $\tilde{\boldsymbol{H}} = \lambda \tilde{\boldsymbol{g}}$, has been adopted so that equation (1b) reduces to

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{b} = \langle \boldsymbol{b}, \dot{\boldsymbol{x}}\rangle \boldsymbol{b} - \tfrac{1}{2}\tilde{\boldsymbol{g}}^\sharp(\boldsymbol{b},\boldsymbol{b})\dot{\boldsymbol{x}}^\flat + \lambda \dot{\boldsymbol{x}}^\flat. \tag{2}$$

In the following analysis we fix the initial conditions for the conformal curve equations such that the curves of the congruence are orthogonal to the initial hypersurface $\mathscr{S}$, while the initial value of the 1-form $\boldsymbol{b}$ is given by $\Omega^{-1}\mathrm{d}\Omega$ where Ω is a conformal factor inducing a 1-point compactification of an asymptotically Euclidean end.

3 Conformal curves in the Reissner-Nordström spacetime

A natural question to be asked is whether conformal geodesics, and more generally, the class of conformal curves introduced in [10] can be used to analyse global aspects of black hole spacetimes. In this respect, in [5] it has been shown that the maximal extension of the Schwarzschild spacetime, the so-called Schwarzschild-Kruskal spacetime [11], can be covered with a congruence of conformal geodesics which has no conjugate points. The conformal Gaussian gauge system obtained using this congruence offers a vantage perspective for the study of conformal properties of the Schwarzschild spacetime and for its global evaluation by means of numerical methods —see e.g. [12].

In the present contribution we briefly review the main results of [10] where the question has been raised to what extent a similar construction can be performed for the Reissner-Nordström spacetime. The consideration of the Reissner-Nordström spacetime is, for several reasons, natural. The inclusion of the electromagnetic field provides a model of angular momentum —see e.g. [13, 14]. We expect our analysis to provide insights into more general (i.e. less symmetric) situations —e.g. the Kerr and Kerr-Newman spacetimes. In addition, there is the expectation that black hole spacetimes with timelike singularities could be more tractable from the point of view of the conformal geometry than black holes with spacelike singularities. This expectation is based on the analysis of the structure of spatial infinity of the Schwarzschild spacetime. In this case the well understood divergence of the Weyl tensor at spatial infinity can also be regarded as the timelike singularity of a negative mass Schwarzschild spacetime —see e.g. [15] for a conformal diagram of this.

The main result of the analysis in [10] is the following:

Theorem 1. *The domains of outer communication of the extremal and the non-extremal Reissner-Nordström spacetime can be covered with a timelike congruence* *of conformal curves which contains no conjugate points. The congruence of conformal curves extends smoothly to null infinity.*

Numerical solutions of the conformal curve equations show, in fact, that conjugate points do arise after the curves have crossed the horizon and entered the black hole region. From the perspective of the Cauchy problem for the Einstein field equations, these conjugate points are not a major concern as one is mainly interested in the behaviour of the spacetime in the domain of outer communication and at the horizon. This is, in particular, the case in the problem of the so-called *non-linear stability of black hole spacetimes* —see e.g. [16].

Our main result provides a suitable conformal gauge to analyse the properties of the Reissner-Nordström spacetime by means of the conformal Einstein field equations. In particular, it opens the possibility of global numerical evaluations of the spacetime [17] similar to the ones carried out in [12] for the Schwarzschild spacetime.

4 Conformal curves in other spacetimes

The analysis of [5] and [10] suggest the possibility of obtaining analogues of Theorem 1 for the Schwarzschild-de Sitter and Schwarzschild-anti de Sitter spacetimes. More generally, one could also consider the Reissner-Nordström-de Sitter/anti-de Sitter spacetimes. These spacetimes are static. In order to consider dynamical situations, while at the same time retaining the assumption of spherical symmetry, one has to bring into play other matter models. A suitable test case is given by the Vaidya solution. Alternatively, one could consider spacetimes arising from the (spherically symmetric) Einstein-scalar field and Einstein-Maxwell-scalar field system.

In general terms, it is conjectured that for spacetimes with a Maxwell field it should be possible to obtain a result similar to that of Theorem 1 where at least the outer domain of communication of the spacetime can be covered without having conjugate points. If no Maxwell field is present then the situation should resemble that of the analysis of the Schwarzschild spacetime in [5] where even the regions inside the black hole can be covered by the congruence.

5 Qualitative behaviour of the curves

For our choice of initial data, the analysis of [10] identified 3 different types of possible behaviours for the conformal curves. First of all, one has curves starting in the asymptotic region of an initial hypersurface $\mathscr{S}$ which extend up to (and including) future null infinity. Null infinity is reached for a finite value of a conformally privileged parameter τ. A different type of behaviour is given by conformal curves starting closer to the horizon. These curves exhibit a periodic behaviour on the conformal diagram of the spacetime, reaching the horizon in a finite value of τ and then

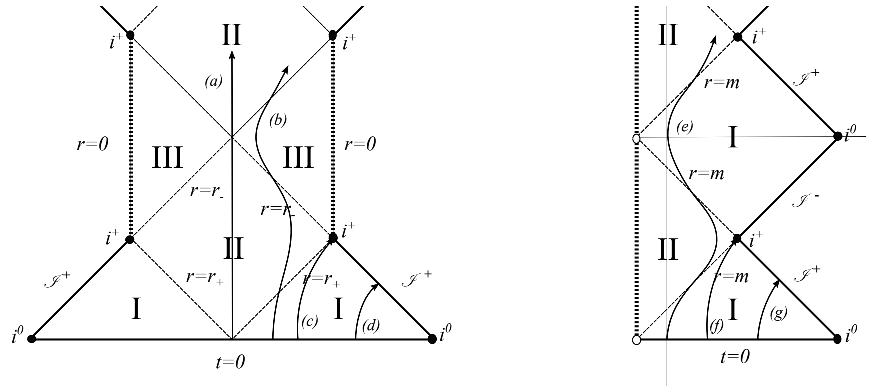

Fig. 1 Schematic illustration of the behaviour of conformal curves . To the left the non-extremal case: (a) the curve starting at $r_* = r_+$; (b) a curve with $r_+ < r_* < r_{\circledast}$; (c) the critical curve $r_* = r_{\circledast}$; (d) a curve with $r_* > r_{\circledast}$. To the right the extremal case: (e) a curve with $r_* < r_{\circledast}$; (f) the critical curve $r_* = r_{\circledast}$; (g) a curve with $r_* > r_{\circledast}$. *The curves are not depicted to scale.*

entering the so-called regions II and III (in the non-extremal case) and the region II (in the extremal case) before re-emerging in a domain of outer communication. An extreme case of this behaviour is that of the conformal curve starting exactly at the bifurcation sphere of the non-extremal case —this corresponds to the curve (a) of Figure 1. A curious property of these curves is that they cannot get arbitrarily close to the singularity. Hence, there are regions inside the black hole region which cannot be probed by means of the curves. Separating the behaviour of curves entering the black hole and those escaping to null infinity, one has *critical curves* which exactly hit timelike infinity (i^+) in a finite amount of the parameter τ. These curves are of special relevance for disentangling the conformal structure of i^+. In particular, the analysis of [10] shows that in the non-extremal case, the critical curves become null at i^+. A similar behaviour had been identified in the Schwarzschild spacetime [4]. Remarkably, for the extremal case the critical conformal curves remain timelike even at i^+.

6 Conclusions

The type of analysis described in this contribution is a first step in the study of the Reissner-Nordström spacetime (and more generally, spherically symmetric spacetimes) as a solution of the conformal field equations. The main conclusion to be extracted is that, at least in what concerns the domain of outer communication, a class of conformally privileged curves can be used to probe the spacetimes. Moreover, the curves can be used individually, not as a congruence, to probe certain regions inside the black hole region. In view of this programme, a result of special relevance is the observation made in [10] that the conformal structure of the timelike infinity

(i^+) of the extremal Reissner-Nordström spacetime may be more tractable, from an analytic point of view, than that of the non-extremal case. With regards to non-linear stability (or lack thereof) of black-hole spacetimes, the fact that the congruence developed conjugate points outside the domain of outer communication poses no real limitation as this is the only region of the spacetime one is really concerned with —see e.g. [16].

Comparing the conformal diagrams of the Reissner-Nordström spacetime and the the Kerr spacetime, it is natural to wonder how much of the structure observed in the present analysis has an analogue in the Kerr solution. For example, it is natural to conjecture that the domain of outer communication of the Kerr spacetime can be covered by means of a non-singular congruence of conformal geodesics reaching beyond null infinity. It is likely that this congruence will degenerate after it has crossed the event horizon and that the curves will have some type of singularity avoiding properties so that there may exists regions in the black hole region which can not be probed by this congruence. A more tantalising possibility is that, as in the case of the extremal Reissner-Nordström spacetime, the extreme Kerr may have a more tractable structure at i^+. In any case, the analysis of conformal geodesics in the Kerr spacetime is bound to be much more complicated as the warped product structure of the line element is lost.

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