Geodesic Deviation in Kundt Spacetimes of any Dimension

Robert Švarc and Jiří Podolský

Abstract Using the invariant form of the equation of geodesic deviation, which describes relative motion of free test particles, we investigate a general family of D-dimensional Kundt spacetimes. We demonstrate that local influence of the gravitational field can be naturally decomposed into Newton-type tidal effects typical for type II spacetimes, longitudinal deformations mainly present in spacetimes of algebraic type III, and type N purely transverse effects corresponding to gravitational waves with $\frac{1}{2}D(D-3)$ independent polarization states. We explicitly study the most important examples, namely exact pp-waves, gyratons, and VSI spacetimes. This analysis helps us to clarify the geometrical and physical interpretation of the Kundt class of nonexpanding, nontwisting and shearfree geometries.

1 Geometry of Kundt spacetimes

The scalars Θ (expansion), A^2 (twist) and σ^2 (shear) characterizing optical properties of an affinely parameterized geodesic null congruence k^a are

$$\Theta = \frac{1}{D-2} k^a_{\ ;a} \,, \quad A^2 = -k_{[a;b]}k^{a;b} \,, \quad \sigma^2 = k_{(a;b)}k^{a;b} - \frac{1}{D-2}(k^a_{\ ;a})^2 \,. \qquad (1)$$

Purely geometric definition of the Kundt family of spacetimes, namely that it admits nonexpanding ($\Theta = 0$), nontwisting ($A = 0$) and shearfree ($\sigma = 0$) such a congruence, implies that there exist suitable coordinates in which the line element of any Kundt spacetime can be written as [1, 2, 3, 4, 5]

$$\mathrm{d}s^2 = g_{ij}(u,x)\,\mathrm{d}x^i\mathrm{d}x^j + 2\,g_{ui}(r,u,x)\,\mathrm{d}x^i\mathrm{d}u - 2\,\mathrm{d}u\,\mathrm{d}r + g_{uu}(r,u,x)\,\mathrm{d}u^2 \,. \qquad (2)$$

Robert Švarc and Jiří Podolský
Institute of Theoretical Physics, Charles University in Prague
V Holešovičkách 2, 180 00 Praha 8, Czech Republic
e-mail: robert.svarc@mff.cuni.cz;podolsky@mbox.troja.mff.cuni.cz

The coordinate r is the affine parameter along the congruence $k^a = \partial_r$, $u = \text{const.}$ are null (wave)surfaces, and $x \equiv (x^2, x^3, \ldots, x^{D-1})$ are $D-2$ spatial coordinates in the transverse Riemannian space. Notice that the spatial part g_{ij} of the metric must be independent of r, all other metric components g_{ui} and g_{uu} can, in principle, be functions of all the coordinates (r, u, x). No specific Einstein field equations have been employed yet.

For such most general Kundt line element (2) a lengthy calculation gives the following components of the Riemann curvature tensor

$$
\begin{aligned}
R_{rprq} &= 0\,, && (3)\\
R_{rpru} &= -\tfrac{1}{2} g_{up,rr}\,, && (4)\\
R_{ruru} &= -\tfrac{1}{2} g_{uu,rr} + \tfrac{1}{4} g^{ij} g_{ui,r} g_{uj,r}\,, && (5)\\
R_{rpkq} &= 0\,, && (6)\\
R_{rpuq} &= \tfrac{1}{2} g_{up,rq} + \tfrac{1}{4} g_{up,r} g_{uq,r} - \tfrac{1}{4} g^{ij} g_{ui,r} \left(2g_{j(p,q)} - g_{pq,j}\right)\,, && (7)\\
R_{rupq} &= g_{u[p,q],r}\,, && (8)\\
R_{ruup} &= g_{u[u,p],r} + \tfrac{1}{4} g^{ri} g_{up,r} g_{ui,r} - \tfrac{1}{4} g^{ij} g_{ui,r} \left(2g_{j(u,p)} - g_{up,j}\right)\,, && (9)\\
R_{kplq} &= {}^{S}R_{kplq}\,, && (10)\\
R_{upkq} &= g_{p[k,q],u} - g_{u[k,q],p} \\
&\quad + \tfrac{1}{4} \left[g_{uk,r} \left(g_{pq,u} - 2g_{u(p,q)} \right) - g_{uq,r} \left(g_{pk,u} - 2g_{u(p,k)} \right) \right] \\
&\quad + \tfrac{1}{4} g^{ri} \left[g_{uk,r} \left(2g_{i(p,q)} - g_{pq,i} \right) - g_{uq,r} \left(2g_{i(p,k)} - g_{pk,i} \right) \right] \\
&\quad + \tfrac{1}{4} g^{ij} \left(2g_{j(u,q)} - g_{uq,j} \right) \left(2g_{i(p,k)} - g_{pk,i} \right) \\
&\quad - \tfrac{1}{4} g^{ij} \left(2g_{j(u,k)} - g_{uk,j} \right) \left(2g_{i(p,q)} - g_{pq,i} \right)\,, && (11)\\
R_{upuq} &= g_{u(p,q),u} - \tfrac{1}{2} \left(g_{pq,uu} + g_{uu,pq} \right) + \tfrac{1}{4} g^{rr} g_{up,r} g_{uq,r} \\
&\quad - \tfrac{1}{4} g_{uu,r} \left[2g_{u(p,q)} - g_{pq,u} - g^{ri} \left(2g_{i(p,q)} - g_{pq,i} \right) \right] \\
&\quad + \tfrac{1}{4} g_{up,r} \left[g_{uu,q} - g^{ri} \left(2g_{i(u,q)} - g_{uq,i} \right) \right] \\
&\quad + \tfrac{1}{4} g_{uq,r} \left[g_{uu,p} - g^{ri} \left(2g_{i(u,p)} - g_{up,i} \right) \right] \\
&\quad + \tfrac{1}{4} g^{ij} \left(2g_{j(u,p)} - g_{up,j} \right) \left(2g_{i(u,q)} - g_{uq,i} \right) \\
&\quad - \tfrac{1}{4} g^{ij} \left(2g_{uj,u} - g_{uu,j} \right) \left(2g_{i(p,q)} - g_{pq,i} \right)\,, && (12)
\end{aligned}
$$

where i, j, k, l, p, q denote the spatial components (and derivatives w.r.t.) x. The superscript "^{S}" labels tensor quantities corresponding to the spatial metric g_{ij}, with derivatives taken only with respect to the coordinates x. The components of the Ricci tensor are

$$
\begin{aligned}
R_{rr} &= 0\,, && (13)\\
R_{rk} &= -\tfrac{1}{2} g_{uk,rr}\,, && (14)\\
R_{ru} &= -\tfrac{1}{2} g_{uu,rr} + \tfrac{1}{2} g^{ri} g_{ui,rr} + \tfrac{1}{2} g^{pq} g_{up,rq} \\
&\quad + \tfrac{1}{2} g^{pq} g_{up,r} g_{uq,r} - \tfrac{1}{4} g^{pq} g^{ij} g_{ui,r} \left(2g_{jp,q} - g_{pq,j} \right)\,, && (15)
\end{aligned}
$$

$$
\begin{aligned}
R_{pq} &= {}^S R_{pq} - g_{u(p,q),r} - \tfrac{1}{2} g_{up,r} g_{uq,r} + \tfrac{1}{2} g^{kl} g_{uk,r} \left(2g_{l(p,q)} - g_{pq,l}\right), && (16)\\
R_{uu} &= -\tfrac{1}{2} g^{rr} g_{uu,rr} - 2g^{ri} g_{u[u,i],r} + \tfrac{1}{2} g^{pq} \left(2g_{up,uq} - g_{pq,uu} - g_{uu,pq}\right)\\
&\quad -\tfrac{1}{2} g^{rp} g^{rq} g_{up,r} g_{uq,r} + \tfrac{1}{2} g^{rr} g^{pq} g_{up,r} g_{uq,r} + \tfrac{1}{2} g^{pq} g^{ri} g_{up,r} (2g_{q(u,i)} - g_{ui,q})\\
&\quad -\tfrac{1}{4} g^{pq} g_{uu,r} \left[2g_{up,q} - g_{pq,u} - g^{ri} \left(2g_{ip,q} - g_{pq,i}\right)\right]\\
&\quad +\tfrac{1}{2} g^{pq} g_{up,r} \left[g_{uu,q} - g^{ri} \left(2g_{i(u,q)} - g_{uq,i}\right)\right]\\
&\quad +\tfrac{1}{4} g^{pq} g^{ij} \left(2g_{j(u,p)} - g_{up,j}\right) \left(2g_{i(u,q)} - g_{uq,i}\right)\\
&\quad -\tfrac{1}{4} g^{pq} g^{ij} \left(2g_{uj,u} - g_{uu,j}\right) \left(2g_{ip,q} - g_{pq,i}\right), && (17)\\
R_{uk} &= -\tfrac{1}{2} g^{rr} g_{uk,rr} - g_{u[u,k],r} + g^{ri} (g_{u[i,k],r} - g_{k[u,i],r})\\
&\quad +g^{pq} (g_{p[k,q],u} - g_{u[k,q],p}) - \tfrac{1}{2} g^{ri} g_{uk,r} g_{ui,r}\\
&\quad +\tfrac{1}{4} g^{pq} g^{ri} \left[4g_{uq,r} g_{k[p,i]} + g_{uk,r} (2g_{i(p,q)} - g_{pq,i})\right]\\
&\quad +\tfrac{1}{4} g^{pq} \left[2g_{up,r} g_{uq,k} - g_{uk,r} (2g_{up,q} - g_{pq,u})\right]\\
&\quad +\tfrac{1}{4} g^{pq} g^{ij} \left(2g_{j(u,q)} - g_{uq,j}\right) \left(2g_{i(p,k)} - g_{pk,i}\right)\\
&\quad -\tfrac{1}{4} g^{pq} g^{ij} \left(2g_{j(u,k)} - g_{uk,j}\right) \left(2g_{ip,q} - g_{pq,i}\right), && (18)
\end{aligned}
$$

and the Ricci scalar curvature of the Kundt spacetime (2) is given by

$$
\begin{aligned}
R = {}^S R + g_{uu,rr} - 2g^{ri} g_{ui,rr} - 2g^{pq} g_{up,rq}&\\
-\tfrac{3}{2} g^{pq} g_{up,r} g_{uq,r} + g^{pq} g^{kl} g_{uk,r} (2g_{lp,q} - g_{pq,l})&. \qquad (19)
\end{aligned}
$$

2 Applying the field equations

So far we have not specified the matter content of the spacetimes. Now, following the approach presented in [4], we can determine the r-dependence of the metric (2) using the Einstein field equations $R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$. Since $R_{rr} = 0$ and $g_{rr} = 0$, there is an obvious restriction on the energy-momentum tensor allowed in the Kundt family, namely $T_{rr} = 0$. *Assuming* $T_{rk} = 0$, we can directly integrate the Einstein equation $R_{rk} = 0$ with (14), yielding g_{uk} *linear* in r. Using the field equation $R_{ru} + \frac{1}{2} R - \Lambda = 8\pi T_{ru}$, this implies that the component T_{ru} must be independent of r. Taking the trace of Einstein's equations we can also determine the r-dependence of g_{uu}: if the trace T of energy-momentum tensor T_{ab} does not depend on the coordinate r, the metric function g_{uu} can only be (at most) *quadratic* in r, see (19). Under these conditions

$$
\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i \mathrm{d}x^j + 2\,(e_i + f_i\, r)\,\mathrm{d}x^i \mathrm{d}u - 2\,\mathrm{d}u\,\mathrm{d}r + (a\,r^2 + b\,r + c)\,\mathrm{d}u^2\,, \qquad (20)
$$

where all the functions g_{ij}, e_i, f_i, a, b and c are independent of r, and are constrained by the specific Einstein equations [4]. In particular, any *vacuum* Kundt metric, possibly with a cosmological constant Λ and/or aligned electromagnetic field, can be written in the form (20).

3 Geodesic deviation in an arbitrary spacetime

In our recent work [6] we demonstrated that the equation of geodesic deviation, which describes relative motion of nearby free test particles, can in *any* D-dimensional spacetime be expressed in the invariant form

$$
\begin{aligned}
\ddot{Z}^{(1)} &= \frac{2\Lambda}{(D-2)(D-1)}\,Z^{(1)} + \Psi_{2S}\,Z^{(1)} + \frac{1}{\sqrt{2}}\,(\Psi_{1T^j} - \Psi_{3T^j})\,Z^{(j)} \\
&\quad + \frac{8\pi}{D-2}\Big[T_{(1)(1)}\,Z^{(1)} + T_{(1)(j)}\,Z^{(j)} - \Big(T_{(0)(0)} + \frac{2\,T}{D-1}\Big)Z^{(1)}\Big], \qquad (21)\\
\ddot{Z}^{(i)} &= \frac{2\Lambda}{(D-2)(D-1)}\,Z^{(i)} - \Psi_{2T^{(ij)}}\,Z^{(j)} + \frac{1}{\sqrt{2}}\,(\Psi_{1T^i} - \Psi_{3T^i})\,Z^{(1)} \\
&\qquad\qquad - \frac{1}{2}\,(\Psi_{0^{ij}} + \Psi_{4^{ij}})\,Z^{(j)} \\
&\quad + \frac{8\pi}{D-2}\Big[T_{(i)(1)}\,Z^{(1)} + T_{(i)(j)}\,Z^{(j)} - \Big(T_{(0)(0)} + \frac{2\,T}{D-1}\Big)Z^{(i)}\Big], \qquad (22)
\end{aligned}
$$

with $i,j = 2,\ldots,D-1$. Here $Z^{(1)},Z^{(2)},\ldots,Z^{(D-1)}$ are spatial components of the separation vector $\boldsymbol{Z} = Z^a\,\boldsymbol{e}_a$ between the test particles in a natural interpretation orthonormal frame $\{\boldsymbol{e}_a\}$ where $\boldsymbol{e}_{(0)} = \boldsymbol{u}$ is the velocity vector of the fiducial test particle ($\boldsymbol{e}_a \cdot \boldsymbol{e}_b = \eta_{ab}$), $\ddot{Z}^{(1)},\ddot{Z}^{(2)},\ldots,\ddot{Z}^{(D-1)}$ are the corresponding relative accelerations, T_{ab} are frame components of the energy-momentum tensor, and the scalars $\Psi_{A^{\cdots}}$ defined as

$$
\begin{aligned}
\Psi_{0^{ij}} &= C_{abcd}\,k^a\,m_i^b\,k^c\,m_j^d\,, & & \\
\Psi_{1T^i} &= C_{abcd}\,k^a\,l^b\,k^c\,m_i^d\,, & \Psi_{1^{ijk}} &= C_{abcd}\,k^a\,m_i^b\,m_j^c\,m_k^d\,,\\
\Psi_{2S} &= C_{abcd}\,k^a\,l^b\,l^c\,k^d\,, & \Psi_{2^{ijkl}} &= C_{abcd}\,m_i^a\,m_j^b\,m_k^c\,m_l^d\,,\\
\Psi_{2T^{ij}} &= C_{abcd}\,k^a\,m_i^b\,l^c\,m_j^d\,, & \Psi_{2^{ij}} &= C_{abcd}\,k^a\,l^b\,m_i^c\,m_j^d\,,\\
\Psi_{3T^i} &= C_{abcd}\,l^a\,k^b\,l^c\,m_i^d\,, & \Psi_{3^{ijk}} &= C_{abcd}\,l^a\,m_i^b\,m_j^c\,m_k^d\,,\\
\Psi_{4^{ij}} &= C_{abcd}\,l^a\,m_i^b\,l^c\,m_j^d\,, & & \qquad (23)
\end{aligned}
$$

$i,j,k,l = 2,\ldots,D-1$, are components of the Weyl tensor with respect to the null frame $\{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}_i\}$ associated with $\{\boldsymbol{e}_a\}$ via the relations $\boldsymbol{k} = \frac{1}{\sqrt{2}}(\boldsymbol{u}+\boldsymbol{e}_{(1)})$, $\boldsymbol{l} = \frac{1}{\sqrt{2}}(\boldsymbol{u}-\boldsymbol{e}_{(1)})$, $\boldsymbol{m}_i = \boldsymbol{e}_{(i)}$, see figure 1.

Components of the Weyl tensor (23) are listed by their boost weight and directly generalize the standard Newman–Penrose complex scalars Ψ_A known from the $D=4$ case [7, 6]. In equations (21), (22), only the "electric part" of the Weyl tensor represented by the scalars in the left column of (23) occurs, and there are various constraints and symmetries, for example

$$
\begin{aligned}
&\Psi_{1T^i} = \Psi_{1^k{}^k{}_i}\,,\ \Psi_{2S} = \tfrac{1}{2}\Psi_{2^{kl}{}^{kl}}\,,\ \Psi_{2T^{(ij)}} = \tfrac{1}{2}\Psi_{2^{ikj}{}^k}\,,\ \Psi_{3T^i} = \Psi_{3^k{}^k{}_i}\,,\\
&\Psi_{0^{ij}} = \Psi_{0^{(ij)}}\,,\quad \Psi_{0^k{}^k} = 0\,,\qquad \Psi_{4^{ij}} = \Psi_{4^{(ij)}}\,,\quad \Psi_{4^k{}^k} = 0\,. \qquad (24)
\end{aligned}
$$

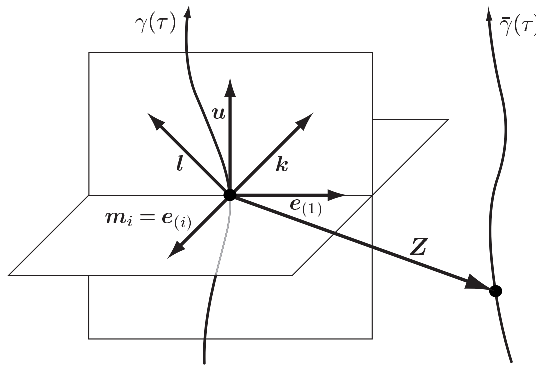

Fig. 1 Evolution of the separation vector $\boldsymbol{Z}$ that connects particles moving along geodesics $\gamma(\tau)$, $\bar{\gamma}(\tau)$ is given by the equation of geodesic deviation (21) and (22). Its components are expressed in the orthonormal frame $\{\boldsymbol{e}_a\}$, $\boldsymbol{e}_{(0)} = \boldsymbol{u}$. The associated null frame $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}_i\}$ is also indicated.

4 Geodesic deviation in Kundt spacetimes

For the general Kundt spacetime (2), the null interpretation frame adapted to an arbitrary observer moving along a timelike geodesic $\gamma(\tau)$ with velocity $\boldsymbol{u} = \dot{r}\,\partial_r + \dot{u}\,\partial_u + \dot{x}^i\partial_i$ takes the form

$$\begin{aligned}
\boldsymbol{k} &= \frac{1}{\sqrt{2}\,\dot{u}}\,\partial_r\,, \quad \boldsymbol{l} = \Big(\sqrt{2}\,\dot{r} - \frac{1}{\sqrt{2}\,\dot{u}}\Big)\partial_r + \sqrt{2}\,\dot{u}\,\partial_u + \sqrt{2}\dot{x}^i\partial_i\,,\\
\boldsymbol{m}_i &= \frac{1}{\dot{u}}\,m_i^j(g_{ju}\,\dot{u} + g_{jk}\,\dot{x}^k)\,\partial_r + m_i^j\,\partial_j\,,
\end{aligned} \tag{25}$$

where m_i^j satisfy $g_{jl}\,m_i^j\,m_k^l = \delta_{ik}$ to fulfil $\boldsymbol{m}_i \cdot \boldsymbol{m}_k = \delta_{ik}$, $\boldsymbol{k}\cdot\boldsymbol{l} = -1$. Vector $\boldsymbol{k}$ is oriented along the nonexpanding, nontwisting and shearfree null congruence $k^a = \partial_r$ defining the Kundt family. Moreover, $\boldsymbol{u} = \frac{1}{\sqrt{2}}(\boldsymbol{k}+\boldsymbol{l})$ and $\boldsymbol{e}_{(1)} = \frac{1}{\sqrt{2}}(\boldsymbol{k}-\boldsymbol{l}) = \sqrt{2}\,\boldsymbol{k} - \boldsymbol{u}$. The spatial vector $\boldsymbol{e}_{(1)}$ is thus uniquely determined by the geometrically privileged null congruence of the Kundt family, and the observer's velocity $\boldsymbol{u}$. For this reason we call such a special direction $\boldsymbol{e}_{(1)}$ *longitudinal*, while the $D-2$ directions $\boldsymbol{e}_{(i)} = \boldsymbol{m}_i$ *transverse*.

In order to evaluate the scalars (23) we need to calculate the Weyl tensor

$$C_{abcd} = R_{abcd} - \frac{2}{D-2}\big(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\big) + \frac{2R\,g_{a[c}\,g_{d]b}}{(D-1)(D-2)}\,, \tag{26}$$

using the components of the Riemann and Ricci tensors (3)–(19). We immediately observe that $C_{rprq} = 0$ which implies $\Psi_{0ij} = 0$. Therefore, *all Kundt spacetimes are of algebraic type* I, or more special, and ∂_r is WAND.

Restricting now to the important subfamily (20) for which

$$g_{ui} = e_i(u,x) + f_i(u,x)\, r\,, \qquad g_{uu} = a(u,x)\, r^2 + b(u,x)\, r + c(u,x)\,, \tag{27}$$

we obtain $R_{rpru} = 0$, $R_{rp} = 0$ which implies $C_{rpru} = 0$, $C_{rpkq} = 0$ so that $\Psi_{1T^i} = 0$, $\Psi_{1^{ijk}} = 0$. Since all Weyl scalars of boost weights 2 and 1 vanish, the metric (20) represents Kundt spacetimes of algebraic type II (or more special). Equations (21), (22) for the geodesic deviation (omitting the frame components of T_{ab} encoding the direct influence of matter) in the case of the Kundt class of spacetimes (20) thus simplify considerably to

$$\begin{aligned}
\ddot{Z}^{(1)} &= \frac{2\Lambda}{(D-2)(D-1)}\, Z^{(1)} + \Psi_{2S}\, Z^{(1)} - \frac{1}{\sqrt{2}}\, \Psi_{3T^j}\, Z^{(j)}\,, \\
\ddot{Z}^{(i)} &= \frac{2\Lambda}{(D-2)(D-1)}\, Z^{(i)} - \Psi_{2T^{(ij)}}\, Z^{(j)} - \frac{1}{\sqrt{2}}\, \Psi_{3T^i}\, Z^{(1)} - \frac{1}{2}\, \Psi_{4^{ij}}\, Z^{(j)}\,,
\end{aligned} \tag{28}$$

where the only nonvanishing Weyl scalars are

$$\begin{aligned}
\Psi_{2S} &= -R_{ruru} + \tfrac{2}{D-2} R_{ru} + \tfrac{1}{(D-1)(D-2)} R\,, \\
\Psi_{2T^{ij}} &= m_i^p m_j^q \big[R_{rpuq} - \tfrac{1}{D-2} \left(g_{pq} R_{ru} - R_{pq} \right) - \tfrac{1}{(D-1)(D-2)} R\, g_{pq} \big]\,, \\
\Psi_{3T^j} &= \sqrt{2}\, m_j^p \Big\{ \dot{x}^k \big[R_{ruru}\, g_{kp} - R_{rkup} - R_{rukp} - \tfrac{1}{D-2} \left(g_{kp} R_{ru} + R_{kp} \right) \big] \\
&\qquad + \dot{u} \big[R_{ruru}\, g_{up} - R_{ruup} - \tfrac{1}{D-2} \left(g_{up} R_{ru} + R_{up} \right) \big] \Big\}\,, \\
\Psi_{4^{ij}} &= 2\, m_{(i}^p m_{j)}^q \Big\{ \dot{x}^k \dot{x}^l \big[R_{rpuq}\, g_{kl} - g_{pk} (2R_{rluq} - g_{lq} R_{ruru} + 2R_{rulq}) \\
&\qquad\qquad + R_{kplq} - \tfrac{1}{D-2} g_{pq} \left(g_{kl} R_{ru} + R_{kl} \right) \big] \\
&\qquad + 2\dot{u}\dot{x}^k \big[R_{rpuq}\, g_{uk} - g_{up} (R_{rkuq} - R_{ruru}\, g_{qk} + R_{rukq}) \\
&\qquad\qquad - R_{ruup}\, g_{qk} + R_{upkq} - \tfrac{1}{D-2} g_{pq} (g_{uk} R_{ru} + R_{uk}) \big] \\
&\qquad + \dot{u}^2 \big[R_{rpuq}\, g_{uu} - g_{uq} \left(2R_{ruup} - g_{up} R_{ruru} \right) + R_{upuq} \\
&\qquad\qquad - \tfrac{1}{D-2} g_{pq} \left(g_{uu} R_{ru} + R_{uu} \right) \big] \Big\}\,,
\end{aligned} \tag{29}$$

and the components R_{abcd} are explicitly given by (5)–(12), R_{ab} by (15)–(18), and the Ricci scalar curvature R is given by (19).

The relative motion of free test particles in any Kundt spacetime (20) is thus composed of the *isotropic influence* of the cosmological constant Λ, *Newton-like tidal deformations* represented by Ψ_{2S}, $\Psi_{2T^{(ij)}}$, *longitudinal* accelerations associated with the direction $+\mathbf{e}_{(1)}$ given by Ψ_{3T^j}, and by *transverse gravitational waves* propagating along $+\mathbf{e}_{(1)}$ encoded in the symmetric traceless matrix $\Psi_{4^{ij}}$, see (24). The invariant amplitudes (29) combine the curvature of the Kundt spacetime with kinematics of the specific geodesic motion. In contrast to longitudinal and transverse wave effects, the Newton-like deformations caused by Ψ_{2S} and $\Psi_{2T^{(ij)}}$ are independent of the observer's velocity components $\dot{x}^i$ and $\dot{u}$.

More details can be found in our recent publications [8, 9].

5 Discussion of particular subfamilies

The Kundt class involves several physically interesting subfamilies, for example pp-waves including gyratons and VSI spacetimes.

The *pp-waves* are defined by admitting a covariantly constant null vector field k^a [2, 3]. They thus belong to the Kundt class with all metric functions independent of r, which is the metric (20) with $f_i = 0, a = 0 = b$:

$$\mathrm{d}s^2 = g_{ij}(u,x)\,\mathrm{d}x^i\mathrm{d}x^j + 2\,e_i(u,x)\,\mathrm{d}x^i\mathrm{d}u - 2\,\mathrm{d}u\,\mathrm{d}r + c(u,x)\,\mathrm{d}u^2\,. \tag{30}$$

The components e_i encode the possible presence of gyratonic matter.

The *VSI spacetimes* have the property that their scalar curvature invariants of all orders vanish identically. As shown in [10], these spacetimes must be of the form (20) with flat transverse space $g_{ij} = \delta_{ij}$:

$$\mathrm{d}s^2 = \delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j + 2\,(e_i + f_i\,r)\,\mathrm{d}x^i\mathrm{d}u - 2\,\mathrm{d}u\,\mathrm{d}r + (a\,r^2 + b\,r + c)\,\mathrm{d}u^2\,. \tag{31}$$

It is straightforward to apply our general results (28) to these particular subcases by evaluating the corresponding Weyl scalars (29) and discussing their specific influence on test particles. We have to restrict ourselves only to the simplest case here,[1] to *vacuum VSI pp-waves without gyratons*:

$$\mathrm{d}s^2 = \delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j - 2\,\mathrm{d}u\,\mathrm{d}r + c(u,x)\,\mathrm{d}u^2\,. \tag{32}$$

Since $\Lambda = 0$, $\Psi_{2S} = 0 = \Psi_{2T^{ij}}$, $\Psi_{3T^j} = 0$, the geodesic deviation reduces to

$$\ddot{Z}^{(1)} = 0\,, \qquad \ddot{Z}^{(i)} = -\tfrac{1}{2}\,\Psi_{4ij}\,Z^{(j)}\,. \tag{33}$$

This clearly represents *gravitational waves* propagating along the spatial direction (1), with the test particles influenced only in the *transverse* directions $(i) = (2),(3),\ldots,(D-1)$. The elements of the *symmetric and traceless* $(D-2)\times(D-2)$ matrix $\Psi_{4ij} = -\dot{u}^2\,c_{,ij}$ (where $\dot{u}$ is a constant) directly encode the corresponding wave amplitudes. Obviously, there are $\frac{1}{2}D(D-3)$ independent *polarization states*.

Taking, e.g., a quadratic function $c(x) \equiv \sum_{i=2}^{D-1} \mathscr{A}_i\,(x^i)^2$ where the constants must satisfy $\sum_{i=2}^{D-1} \mathscr{A}_i = 0$, the wave-amplitude matrix becomes $\Psi_{4ij} = -2\dot{u}^2\,\mathrm{diag}(\mathscr{A}_2, \mathscr{A}_3, \ldots)$. Relative motion of (initially static) particles given by (33) can be explicitly integrated: in the spatial directions with *positive eigenvalues* $\mathscr{A}_i > 0$ they *recede* as $Z^{(i)}(\tau) = Z_0^{(i)} \cosh\left(\sqrt{\mathscr{A}_i}\,|\dot{u}|\,\tau\right)$, while with *negative eigenvalues* $\mathscr{A}_i < 0$ they *converge* as $Z^{(i)}(\tau) = Z_0^{(i)} \cos\left(\sqrt{\mathscr{A}_i}\,|\dot{u}|\,\tau\right)$, and in the directions where $\mathscr{A}_i = 0$ the particles stay fixed, $Z^{(i)}(\tau) = Z_0^{(i)}$.

In principle, the presence of higher-dimensional components of gravitational waves could be observed by detectors in our (1+3)-dimensional universe as the *violation of the standard TT-property*. Indeed, taking the simplest case $D = 5$, the ma-

[1] A thorough discussion of other cases will follow in our subsequent paper.

trix reads $\Psi_{4ij} = -2\dot{u}^2\,\mathrm{diag}(\mathscr{A}_2, \mathscr{A}_3, \mathscr{A}_4)$ where $\mathscr{A}_2 = -(\mathscr{A}_3 + \mathscr{A}_4)$. In the absence of the higher-dimensional component, $\mathscr{A}_4 = 0$, an interferometer in our space detects usual deformations shown in the left part of figure 2. But if $\mathscr{A}_4 \neq 0$ then $\mathscr{A}_2 \neq -\mathscr{A}_3$, and a peculiar deformation, such as on the right part of figure 2, would be observed.

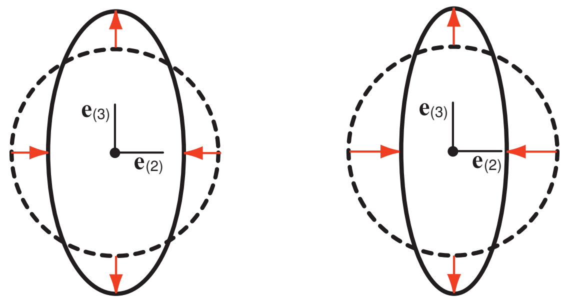

Fig. 2 Standard (left) and one of peculiar deformations of a detector indicating extension of the gravitation wave into higher dimensions (right).

Acknowledgement. R.Š. was supported by the grants GAČR 202/09/0772 and SVV-267301, and J.P. by the grants GAČR P203/12/0118 and MSM0021620860.

References

1. W. Kundt, *Exact solutions of the field equations: Twist-free pure radiation fields*, Proc. R. Soc. London, Ser. A **270**, 328 (1962)
2. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, *Exact Solutions of Einstein's Field Equations*, 2nd edn. Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 2003)
3. J.B. Griffiths, J. Podolský, *Exact Space-Times in Einstein's General Relativity*. Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge; New York, 2012)
4. J. Podolský, M. Žofka, *General Kundt spacetimes in higher dimensions*, Class. Quant. Grav. **26**, 105008 (2009)
5. A. Coley, S. Hervik, G. Papadopoulos, N. Pelavas, *Kundt spacetimes*, Class. Quantum Grav. **26**, 105016 (2009)
6. J. Podolský, R. Švarc, *Interpreting spacetimes of any dimension using geodesic deviation*, Phys. Rev. D **85**, 044057 (2012)
7. P. Krtouš, J. Podolský, *Asymptotic structure of radiation in higher dimensions*, Class. Quantum Grav. **23**, 1603 (2006)
8. J. Podolský, R. Švarc, *Explicit algebraic classification of Kundt geometries in any dimension*, Classical and Quantum Gravity **30**, 125007 (2013)
9. J. Podolský, R. Švarc, *Physical interpretation of Kundt spacetimes using geodesic deviation*, Classical and Quantum Gravity **30**, 205016 (2013)
10. A. Coley, A. Fuster, S. Hervik, N. Pelavas, *Higher dimensional VSI spacetimes*, Class. Quant. Grav. **23**, 7431 (2006)