Scalar Averaging in Szekeres Models

Roberto A. Sussman

Abstract We introduce a formalism of scalar proper volume weighted averages (the "q–average") acting on compact comoving domains of quasi–spherical Szekeres models with a dust source. Although these models do not admit isometries, the resulting averaged scalars are spherically symmetric functionals that can be defined as local functions (the "q–scalars") by considering a varying averaging domain. The fluctuations of the density and Hubble scalar with respect to their corresponding q–scalars determine the Riemann, Weyl, electric Weyl and shear tensors through irreducible covariant algebraic expansions. The q–average of all invariant scalars formed by contractions of these tensors are directly related to statistical variance and covariance moments of the density and Hubble scalar with respect to their q–averages. The q–scalars and q–averages, together with their fluctuations, lead to complete systems of evolution equations and algebraic constraints that fully determine the dynamics of the models. However, these evolution equations lack the "back–reaction" correlation terms characteristic of Buchert's averaging scheme.

1 Introduction

The Szekeres dust models are a well known class of exact solution of Einstein's equations that admit (in general) no Killing vectors [1]. For this reason they are natural candidates to construct models of cosmological inhomogeneities that are much less idealized than spherically symmetric configurations that arise from the Lemaître–Tolman–Bondi (LTB) models [2] (their spherical limiting case). However, Szekeres models provide also an ideal theoretical framework for the study of generic properties of inhomogeneity, such as averaging. In this brief article we introduce a formalism of proper volume weighted averages acting on compact comoving

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domains, and thus we consider only the "quasi–spherical" QS subclass of models for which such domains always exist [3].

2 Quasi–spherical Szekeres models

QS Szekeres dust models can be described by the following metric reminiscent of the spherically symmetric LTB metric:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\mathscr{E}^2 Y'^2}{1-K}\mathrm{d}r^2 + Y^2\left[\mathrm{d}x^2+\mathrm{d}y^2\right], \tag{1}$$

where $Y = Y(t,r,x,y)$ and $\mathscr{E} = \mathscr{E}(r,x,y)$ are given by

$$Y = \frac{R}{\mathscr{E}}, \qquad \mathscr{E} = \frac{S}{2}\left[1+\left(\frac{x-P}{S}\right)^2+\left(\frac{x-Q}{S}\right)^2\right], \tag{2}$$

with $R = R(t,r)$ being the area distance that appears in LTB metrics and $S(r)$, $P(r)$, $Q(r)$ are the Szekeres characteristic free arbitrary functions, so that (1) becomes the LTB metric if they are constants. The metric function Y satisfies an analogous Friedman–like equation as R in LTB models:

$$\dot{Y}^2 = \frac{2\tilde{M}}{Y} - \tilde{K}, \qquad \tilde{M} = \frac{M(r)}{\mathscr{E}^3}, \qquad \tilde{K} = \frac{K(r)}{\mathscr{E}^2}. \tag{3}$$

The main covariant scalars are the density, ρ, the Hubble expansion scalar $\Theta = \nabla_a u^a$ and the Ricci scalar ${}^3\mathscr{R}$ of hypersurfaces ${}^3\mathscr{T}[t]$ orthogonal to u^a take also LTB–like form:

$$\frac{4\pi}{3}\rho = \frac{\tilde{M}'}{3Y^2Y'}, \qquad \mathscr{H} \equiv \frac{\Theta}{3} = \frac{(Y^2Y')^{\cdot}}{3Y^2Y'}, \qquad \mathscr{K} \equiv \frac{{}^3\mathscr{R}}{6} = \frac{(\tilde{K}Y)'}{3Y^2Y'}. \tag{4}$$

3 Quasi–local average and quasi–local scalars.

Let A be a scalar function defined along an arbitrary hypersurface ${}^3\mathscr{T}[t]$ whose proper volume element is $\mathrm{d}\mathscr{V} = \mathscr{F}^{-1}Y^2Y'\mathrm{d}r\mathrm{d}x\mathrm{d}y$, with $\mathscr{F} \equiv \sqrt{1-K}/\mathscr{E}$, the quasi–local scalar average of A for a compact comoving domain $\mathscr{D}[r_b]$ bounded by $r = r_b$ is the linear functional

$$\langle A\rangle_q[r_b] = \frac{\int_{\mathscr{D}} A\mathscr{F}\mathrm{d}\mathscr{V}}{\int_{\mathscr{D}}\mathscr{F}\mathrm{d}\mathscr{V}} = \frac{\int\mathrm{d}y\int\mathrm{d}x\int_0^{r_b} AY^2Y'\mathrm{d}r}{\int\mathrm{d}y\int\mathrm{d}x\int_0^{r_b} Y^2Y'\mathrm{d}r}, \tag{5}$$

which applied to $A = \rho, \mathscr{H}, \mathscr{K}$ yields (with the help form (3) and (4)) averaged quantities that do not depend on the "non–spherical" coordinates (x,y) (even if the

A depend on all 4 coordinates, see [3, 4]):

$$\frac{4\pi}{3}\langle\rho\rangle_q[r_b] = \frac{\tilde{M}_b}{Y_b^3} = \frac{M_b}{R_b^3}, \qquad \langle\mathscr{K}\rangle_q[r_b] = \frac{\tilde{K}_b}{Y_b^2} = \frac{K_b}{R_b^2}, \tag{6}$$

$$\langle\mathscr{H}\rangle_q[r_b] = \frac{\dot{Y}_b}{Y_b} = \frac{\dot{R}_b}{R_b}, \qquad \langle\mathscr{H}\rangle_q^2[r_b] = \frac{8\pi}{3}\langle\rho\rangle_q[r_b] - \langle\mathscr{K}\rangle_q[r_b], \tag{7}$$

where the subindex $_b$ denotes evaluation at $r = r_b$. Since r_b is arbitrary, we can construct the following functions of (t, r) that evaluate locally (the "q–scalars") from the functionals (6)–(7) by considering domains with varying boundary:

$$\frac{4\pi}{3}\rho_q = \frac{M}{R^3}, \quad \mathscr{H}_q = \frac{\dot{R}}{R}, \quad \mathscr{K}_q = \frac{K}{R^2}, \qquad \mathscr{H}_q^2 = \frac{8\pi}{3}\rho_q - \mathscr{K}_q. \tag{8}$$

The relevant curvature and kinematic tensors of the models: the Riemann ($\mathscr{R}^{ab}_{cd}$), Ricci ($\mathscr{R}_{ab}$), Weyl (C^{acbd}) and electric Weyl ($E^{ab} = u_c u_d C^{acbd}$) tensors, as well as the shear tensor (σ_{ab}), are all expressible in terms of irreducible algebraic decompositions containing only the metric, the projection tensor $h_{ab} = g_{ab} + u_a u_b$ and a common divergence–less tensor $\mathbf{e}_{ab} = h_{ab} - 3\eta_a\eta_b$ ($\eta_a = \sqrt{h_{rr}}\delta^r_a$), with the coefficients given by ρ, $\mathscr{H}$ and their fluctuations with respect to ρ_q, $\mathscr{H}_q$:

$$\mathscr{R}^{ab}_{cd} = \frac{8\pi}{3}\rho\left(3\delta^{[a}_{[c}\delta^{b]}_{d]} + 6\delta^{[a}_{[c}u^{b]}u_{d]} - \delta^{a}_{[c}\delta^{b}_{d]}\right) + C^{ab}_{cd}, \tag{9}$$

$$\mathscr{R}^a_b = 4\pi\rho\left(h^a_b + u^a u_b\right), \quad C^{ab}_{cd} = \frac{4\pi}{3}\mathbf{D}_q(\rho)(h^{[a}_{[c} - 3u_{[c}u^{[a})\mathbf{e}^{b]}_{d]}), \tag{10}$$

$$\sigma_{ab} = -\mathbf{D}_q(\mathscr{H})\,\mathbf{e}_{ab}, \quad E_{ab} = \frac{4\pi}{3}\mathbf{D}_q(\rho)\mathbf{e}_{ab}, \tag{11}$$

where the fluctuations $\mathbf{D}_q(\rho)$ and $\mathbf{D}_q(\mathscr{H})$ are defined as

$$\frac{4\pi}{3}\mathbf{D}_q(\rho) = \frac{4\pi}{3}(\rho - \rho_q) = \frac{4\pi}{3}\frac{\rho_q'}{3Y'/Y} = \Psi_2, \tag{12}$$

$$\mathbf{D}_q(\mathscr{H}) = \mathscr{H} - \mathscr{H}_q = \frac{\mathscr{H}_q'}{3Y'/Y} = -\Sigma, \tag{13}$$

with Σ and the conformal invariant Ψ_2 (the eigenvalues of E_{ab} and σ_{ab} in terms of $\mathbf{e}_{ab}$) given by

$$\Sigma = \sigma_{ab}\mathbf{e}^{ab} = -\frac{1}{3}\left(\frac{\dot{Y}'}{Y'} - \frac{\dot{Y}}{Y}\right), \qquad \Psi_2 = E_{ab}\mathbf{e}^{ab} = \frac{\tilde{M}}{Y^3} - \frac{4\pi}{3}\rho. \tag{14}$$

4 Statistical fluctuations and invariant scalars.

The statistical fluctuation of a Szekeres scalar A with respect to its q–average $\langle A\rangle_q$ is given by

$$\mathbf{D}^{\text{st}}_q(A) = A(t,r,x,y) - \langle A\rangle_q[r_b](t) \quad \Rightarrow \quad \langle \mathbf{D}^{\text{st}}(A)\rangle_q[r_b] = 0, \tag{15}$$

and is a non–local quantity that depends in inner points of the domain and also on its boundary $r = r_b$. Evidently, the fluctuations $\mathbf{D}_q(\rho)$ and $\mathbf{D}_q(\mathscr{H})$ in (12) and (13) are not statistical fluctuations, as they are evaluated locally and thus $\langle \mathbf{D}_q(\rho)\rangle_q[r_b] \neq 0$ and $\langle \mathbf{D}_q(\mathscr{H})\rangle_q[r_b] \neq 0$ hold. However, as proven in [5, 6], the averages of local quadratic fluctuations coincides with the average of quadratic statistical fluctuations, and thus relates these averages with the variance and covariance statistical moments $\mathbf{Var}_q$ and $\mathbf{Cov}_q$ [1]

$$\langle\, [\mathbf{D}_q(\rho)]^2\,\rangle_q = \langle\, [\mathbf{D}^{\text{st}}_q(\rho)]^2\,\rangle_q = \langle \rho^2\rangle_q - \langle\rho\rangle_q^2 = \mathbf{Var}_q(\rho), \tag{16}$$

$$\langle\, [\mathbf{D}_q(\mathscr{H})]^2\,\rangle_q = \langle\, [\mathbf{D}^{\text{st}}_q(\mathscr{H})]^2\,\rangle_q = \langle \mathscr{H}^2\rangle_q - \langle\mathscr{H}\rangle_q^2 = \mathbf{Var}_q(\mathscr{H}), \tag{17}$$

$$\begin{aligned}\langle\, \mathbf{D}_q(\rho)\,\mathbf{D}_q(\mathscr{H})\,\rangle_q &= \langle\, \mathbf{D}^{\text{st}}_q(\rho)\,\mathbf{D}^{\text{st}}_q(\mathscr{H})\,\rangle_q = \langle \rho\,\mathscr{H}\rangle_q - \langle\rho\rangle_q\langle\mathscr{H}\rangle_q \\ &= \mathbf{Cov}_q(\rho,\mathscr{H}),\end{aligned} \tag{18}$$

where we omitted the domain indicator $[r_b]$ to simplify the notation.

The relation between the local fluctuations $\mathbf{D}_q(\rho), \mathbf{D}_q(\mathscr{H})$ and the covariant scalars Ψ_2 and Σ in (12) and (13) illustrates an interesting and appealing relation between the q–average and the characteristic tensors of the models through the properties (16), (17) and (18): the q–averages of all quadratic contractions of the curvature and shear tensors in (9)–(11) are directly expressible in terms of statistical moments of ρ and $\mathscr{H}$ with respect to $\langle\rho\rangle_q$ and $\langle\mathscr{H}\rangle_q$:

$$\langle \sigma_{ab}\sigma^{ab}\rangle_q = 6\langle \Sigma^2\rangle_q = 6\mathbf{Var}_q(\mathscr{H}), \tag{19}$$

$$\langle E_{ab}E^{ab}\rangle_q = 6\langle(\Psi_2)^2\rangle_q = \frac{32\pi^2}{3}\mathbf{Var}_q(\rho), \tag{20}$$

$$\langle \sigma_{ab}E^{ab}\rangle_q = 6\langle \Sigma\,\mathscr{E}\rangle_q = 8\pi\mathbf{Cov}_q(\rho,\mathscr{H}), \tag{21}$$

$$\langle C_{abcd}C^{abcd}\rangle_q = \frac{256\pi^2}{3}\,\mathbf{Var}_q(\rho) = \frac{4}{3}\,\mathbf{Var}_q(\mathscr{R}) = 8\langle E_{ab}E^{ab}\rangle_q, \tag{22}$$

$$\langle \mathscr{R}_{abcd}\mathscr{R}^{abcd}\rangle_q = \frac{256\pi^2}{3}\left[\mathbf{Var}_q(\rho) + \frac{5}{4}\langle\rho^2\rangle_q\right] = \frac{4}{3}\mathbf{Var}_q(\mathscr{R}) + \frac{5}{3}\langle\mathscr{R}^2\rangle_q, \tag{23}$$

where we used the fact that the Ricci scalar is $\mathscr{R} = 8\pi\rho$, and thus: $\langle\mathscr{R}\rangle_q = 8\pi\langle\rho\rangle_q$ and $\mathscr{R}_{ab}\mathscr{R}^{ab} = \mathscr{R}^2$. We can express the ratio of Weyl vs Ricci curvature ($\Psi_2/\mathscr{R}$) and anisotropic vs isotropic expansion ($\Sigma/\mathscr{H}$) as ratios between the q-scalars ρ_q, $\mathscr{H}_q$ and their counterparts ρ, $\mathscr{H}$:

[1] This result was proven for LTB models, but it is straightforward to prove that it also holds for the QS Szekeres models.

$$\frac{6\Psi_2}{\mathscr{R}} = 1 - \frac{\rho_q}{\rho}, \qquad \frac{\Sigma}{\mathscr{H}} = \frac{\mathscr{H}_q}{\mathscr{H}} - 1, \tag{24}$$

Also, the quadratic ratio of Weyl to Ricci curvatures is expressible as the ratio of the averages of $(\Psi_2)^2$ and $\mathscr{R}^2$, and as a sort of "standard deviation" of ρ with respect to $\langle\rho\rangle_q$:

$$\frac{6\langle E_{ab}E^{ab}\rangle_q}{\langle\mathscr{R}_{ab}\mathscr{R}^{ab}\rangle_q} = \frac{6\langle(\Psi_2)^2\rangle_q}{\langle(\mathscr{R})^2\rangle_q} = \frac{6\mathbf{Var}_q(\rho)}{\langle\rho^2\rangle_q} = \frac{\langle\rho^2\rangle_q - \langle\rho\rangle_q^2}{\langle\rho^2\rangle_q}, \tag{25}$$

A similar standard deviation of $\mathscr{H}$ with respect to $\langle\mathscr{H}\rangle_q$ follows as the quotient of averages of quadratic covariant scalars $\sigma_{ab}\sigma^{ab}$ and $\mathscr{H}^2$:

$$\frac{\langle\sigma_{ab}\sigma^{ab}\rangle_q}{6\langle\mathscr{H}^2\rangle_q} = \frac{\langle\Sigma^2\rangle_q}{\langle\mathscr{H}^2\rangle_q} = \frac{\mathbf{Var}_q(\mathscr{H})}{\langle\mathscr{H}^2\rangle_q} = \frac{\langle\mathscr{H}^2\rangle_q - \langle\mathscr{H}\rangle_q^2}{\langle\mathscr{H}^2\rangle_q}, \tag{26}$$

where we used (14).

5 Comparison with Buchert's average.

Buchert's scalar average is the standard proper volume average, $\langle A\rangle_p[r_b]$, hence it is defined by (5) with $\mathscr{F} = 1$:

$$\langle A\rangle_p[r_b] = \frac{\int_{\mathscr{D}} A\mathrm{d}\mathscr{V}}{\int_{\mathscr{D}}\mathrm{d}\mathscr{V}} = \frac{\int\mathrm{d}y\int\mathrm{d}x\int_0^{r_b} A\mathscr{F}^{-1}Y^2Y'\mathrm{d}r}{\int\mathrm{d}y\int\mathrm{d}x\int_0^{r_b}\mathscr{F}^{-1}Y^2Y'\mathrm{d}r}. \tag{27}$$

Evidently, the scalars Σ and Ψ_2 are not related to the the local fluctuations $\mathbf{D}_p(\rho)$, $\mathbf{D}_p(\mathscr{H})$ (analogous to $\mathbf{D}_p(\rho)$, $\mathbf{D}_p(\mathscr{H})$) through a closed and straightforward manner as (12)–(13). Therefore, the relations between the Buchert's averages of invariant quadratic scalars and the variance and covariance moments with respect to this average is much more complicated than the simple elegant relations (19)–(23). Likewise, we cannot express with this average the ratio of Weyl to Ricci curvature as in (25) and (26).

It is straightforward to show that the "back–reaction" correlations terms that appear when applying Buchert's average to evolution equations vanish if we apply the q–average (5). The Raychaudhuri equation for Szekeres models is

$$\dot{\mathscr{H}} = -\mathscr{H}^2 - \frac{\kappa}{6}\rho - 2\Sigma^2, \tag{28}$$

averaging on both sides, using (13) and the commutation rule (we omit the domain indicator)

$$\frac{\partial}{\partial t}\langle A\rangle_q - \left\langle\frac{\partial A}{\partial t}\right\rangle_q = \langle A\rangle\dot{}_q - \langle\dot{A}\rangle_q = 3\langle\mathscr{H}A\rangle_q - 3\langle\mathscr{H}\rangle_p\langle A\rangle_q, \tag{29}$$

for $A = \mathscr{H}$, we obtain

$$\langle \mathscr{H} \rangle\dot{}_q[r_b] = -\langle \mathscr{H} \rangle_q^2[r_b] - \frac{4\pi}{3}\langle \rho \rangle_q[r_b] + 2\mathscr{Q}_q[r_b], \tag{30}$$

where $\mathscr{Q}_q[r_b]$ is the back–reaction term :

$$\begin{aligned} \mathscr{Q}_q[r_b] &\equiv \langle (\mathscr{H} - \langle \mathscr{H} \rangle_q)^2 \rangle_q - \langle (\mathscr{H} - \mathscr{H}_q)^2 \rangle_q \\ &= \langle \big[\mathbf{D}_q^{\text{st}}(\mathscr{H})\big]^2 \rangle_q[r_b] - \langle [\mathbf{D}_q(\mathscr{H})]^2 \rangle_q[r_b] = 0, \end{aligned} \tag{31}$$

which vanishes identically for every domain as a consequence of (16) (see [5, 6]). Hence, (30) reduces exactly to a FLRW Raychaudhuri equation given in terms of q–averages. While Buchert's average satisfies the same commutation rule (29), the scalar Σ^2 is not directly related to a local fluctuation of $\mathscr{H}$ (*i.e.* as in the relation (13)), hence its application to the Raychaudhuri equation (28) yields a different result: equation (30) with the $_p$ average, but with nonzero back–reaction:

$$\begin{aligned} \mathscr{Q}_p[r_b] &\equiv \langle (\mathscr{H} - \langle \mathscr{H} \rangle_p)^2 \rangle_p - \langle (\mathscr{H} - \mathscr{H}_q)^2 \rangle_p \\ &= \langle \big[\mathbf{D}_p^{\text{st}}(\mathscr{H})\big]^2 \rangle_p[r_b] - \langle \big[\mathbf{D}_q^{\text{st}}(\mathscr{H})\big]^2 \rangle_p[r_b] \neq 0, \end{aligned} \tag{32}$$

where we used (17).

6 Evolution equations.

While the re–interpretation of the dynamics through the presence of extra back–reaction terms is not possible with q–average, the latter yields evolution equations that are complete and self–consistent, as opposed to the evolution equations from Buchert's average that require making extra assumptions on the back–reaction terms in order to close the system. As shown in [3], if we define relative fluctuations (or "exact perturbations") as

$$\Delta^{(A)} \equiv \frac{\mathbf{D}_q(A)}{A_q} = \frac{A - A_q}{A_q} = \frac{A_q'/A_q}{3Y'/Y}, \qquad A = \rho, \mathscr{H}, \mathscr{K}, \tag{33}$$

the dynamics of the models becomes completely determined by the following system of autonomous evolution equations

$$\dot{\rho}_q = -3\rho_q \mathscr{H}_q, \tag{34}$$

$$\dot{\mathscr{H}}_q = -\mathscr{H}_q^2 - \frac{4\pi}{3}\rho_q, \tag{35}$$

$$\dot{\Delta}^{(\rho)} = -3(1 + \Delta^{(\rho)})\,\mathscr{H}_q \Delta^{(\mathscr{H})}, \tag{36}$$

$$\dot{\Delta}^{(\mathscr{H})} = -(1 + 3\Delta^{(\mathscr{H})})\,\mathscr{H}_q \Delta^{(\mathscr{H})} + \frac{4\pi\rho_q}{3\mathscr{H}_q}(\Delta^{(\mathscr{H})} - \Delta^{(\rho)}), \tag{37}$$

subjected to the algebraic constraints

$$\mathscr{H}_q^2 = \frac{8\pi}{3}\rho_q - \mathscr{K}_q, \qquad 2\Delta^{(\mathscr{H})} = \Omega_q\,\Delta^{(\rho)} + (1-\Omega_q)\,\Delta^{(\mathscr{K})}, \tag{38}$$

where we have introduced the following q–scalar analogous to a FLRW Omega factor

$$\Omega_q \equiv \frac{8\pi\rho_q}{3\mathscr{H}_q^2}, \qquad \Omega_q - 1 = \frac{\mathscr{K}_q}{\mathscr{H}_q^2}. \tag{39}$$

If we consider instead non–local statistical relative fluctuations

$$\Delta_{\text{NL}}^{(A)} = \frac{\mathbf{D}_q^{\text{st}}(A)}{\langle A\rangle_q[r_b]}, \tag{40}$$

such that

$$1 + \Delta_{\text{NL}}^{(A)} = \frac{A_q(r)}{\langle A\rangle_q[r_b]}(1+\Delta^{(A)}), \tag{41}$$

we obtain the following system of evolution equations:

$$\langle \mathscr{H}\rangle\dot{}_q[r_b] = -\langle \mathscr{H}\rangle_q^2[r_b] - \frac{4\pi}{3}\langle\rho\rangle_q[r_b], \tag{42}$$

$$\langle \rho\rangle\dot{}_q[r_b] = -3\langle \mathscr{H}\rangle_q[r_b]\langle\rho\rangle_q[r_b], \tag{43}$$

$$\dot{\Delta}_{\text{NL}}^{(\rho)} = -3(1+\Delta_{\text{NL}}^{(\rho)})\,\langle \mathscr{H}\rangle_q[r_b]\,\Delta_{\text{NL}}^{(\mathscr{H})}, \tag{44}$$

$$\begin{aligned}\dot{\Delta}_{\text{NL}}^{(\mathscr{H})} &= -(1+3\Delta_{\text{NL}}^{(\mathscr{H})})\,\langle \mathscr{H}\rangle_q[r_b]\,\Delta_{\text{NL}}^{(\mathscr{H})} + \frac{4\pi\,\langle\rho\rangle_q[r_b]}{3\langle \mathscr{H}\rangle_q^2[r_b]}(\Delta_{\text{NL}}^{(\mathscr{H})} - \Delta_{\text{NL}}^{(\rho)}) \\ &\quad -2\langle \mathscr{H}\rangle_q[r_b]\left(1-\frac{\mathscr{H}_q(r)}{\langle \mathscr{H}\rangle_q[r_b]}\right)^2 + 4(\mathscr{H}_q(r) - \langle \mathscr{H}\rangle_q[r_b])\Delta_{\text{NL}}^{(\mathscr{H})},\end{aligned} \tag{45}$$

which, in order to render a fully complete system, must be supplemented by the evolution equations (34) and (35) for ρ_q and $\mathscr{H}_q$. This system is subjected to the same algebraic constraints (38), but given in terms of q–averages and non–local relative fluctuations. Evidently, (42)–(45) is much more complicated than (34)–(37), and both systems coincide for comoving observers at the domain boundary $r = r_b$ where $\langle \mathscr{H}\rangle_q[r_b] = \mathscr{H}_q(r_b)$ holds for all t.

7 Conclusions.

Szekeres models provide an ideal tool to explore the theoretical consequences of non–trivial inhomogeneity, and in particular, the relation between averaging and the geometric objects that characterize inhomogeneous spacetimes. We have shown how a suitable weighted scalar average (the q–average) allows us to relate the average of invariant scalars and statistical variance and covariance moments of the density and Hubble scalar. We have also shown that the dynamics of the models can be completely determined by evolution equations constructed with these averaged scalars (and functions constructed with them) and their fluctuations, which can be local or non–local (statistical). While the evolution equations of the q–averages lack "back–reaction" terms characteristic of Buchert's average (the q–average with unit weight factor), these evolution equations are complete and self-consistent systems that can be handled numerically in the multiple potential applications of the Szekeres quasi–spherical solution to model building in Cosmology and General Relativity.

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