The Twin Paradox in Static Spacetimes and Jacobi Fields

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Abstract The twin paradox of special relativity formulated in the geometrical setting of general relativity gives rise to the problem of determining the longest timelike curve between a given pair of points. As a first step one solves the local problem for a bundle of nearby curves (geodesics) in terms of Jacobi fields and conjugate points. These provide important information about geometrical properties of the given spacetime. The second step, to determine the globally maximal length curve in the set of all timelike curves with common endpoints, is harder and may be effectively performed only in spacetimes with high symmetries.

1 Formulation of the problem

The twin paradox in special relativity may be considered on two levels of comprehending. The first, lowest level of understanding deals with the problem of why the effect is asymmetric at all: why one twin turns out younger than the other whereas the time dilation effect for two clocks in uniform relative motion is actually symmetric. Most textbooks on special relativity explain the problem on this level by discussing the simplest (traditional) version of the paradox in which one twin remains all the time at rest in one inertial frame and the other uniformly moves to a distant star, then suddenly turns back and returns again at a constant velocity. By considering the hyperplanes of simultaneity of the astronaut during his travel back and forth one shows that the twin at rest must be older at the reunion than the astronaut; however such calculation can be effectively performed only in this simplest case of twins' motions. In consequence it does not provide a deeper understanding of the paradox. In fact, what does occur if both the twins move at non-uniform velocities?

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The generic form of the paradox may be elucidated only on the second level which requires to go beyond the elementary algebra of the special Lorentz transformation. The solution is well known to the expert on relativity (though it is rather infrequently stressed in textbooks) and is based on the identification of physical time measured by a moving clock = proper time = length of the moving clock's worldline. The problem is then reduced to a purely geometrical one of calculating worldline lengths in Minkowski spacetime. Let the twins A and B travel at arbitrary velocities measured in some inertial frame from point (event) P to Q, then their common age at P will increase at Q by

$$s_A = \int_P^Q ds_A = c\int_{t_1}^{t_2} \sqrt{1-\left(\frac{\mathbf{v}_A}{c}\right)^2}\, dt,$$

$$s_B = \int_P^Q ds_B = c\int_{t_1}^{t_2} \sqrt{1-\left(\frac{\mathbf{v}_B}{c}\right)^2}\, dt \neq s_A$$

respectively. Here t_1 and t_2 are the time coordinates of the points in the inertial frame. In the geometrical setting the effect of different ageing of the twins is obvious. Coming back to the simplest version of the paradox, what is less obvious and is rather surprising at first glance as it stands in contradiction to our experience in Euclidean geometry, is that the twin at rest gets older than the twin moving on a curved (accelerated) worldline. This is due to the reverse triangle inequality which gives rise to the theorem that in Minkowski spacetime the timelike straight line is the longest timelike curve between any pair of its points. Yet a curved timelike line joining two points may be arbitrarily small and it makes no sense to ask of how to move from P to (chronologically related) Q in order to use as little as possible of proper time —the interval $s(P,Q)$ may be arbitrarily close to zero. The properly posed problem is which timelike worldline from P to Q has the largest length and in flat spacetime is clear.

The geometrical twin paradox becomes much more interesting in curved spacetimes since the variety of possible physically relevant motions is much greater than in the flat case. Besides comparing lengths of concrete worldlines in a given spacetime one may ask if there are whole classes of worldlines which are longer than curves in other classes and, first of all, which timelike curves attain the maximal length. The first question is relevant in static spacetimes: what makes one twin younger than the other—velocity (with respect to a static observer) or acceleration (acting e.g. on the twin at rest)? Special cases investigated in first works on the subject do not allow one to infer general statements. For instance, Abramowicz, Bajtlik and Kluzniak [1], [2] investigated static worldlines and circular geodesics in Schwarzschild spacetime and concluded that 'in all situations in which the absolute motion may be defined in terms of some invariant global properties of the spacetime, the twin who moves faster with respect to the global standard of rest is younger at the reunion, irrespectively to twins' accelerations'. This conclusion is, however, false already in Schwarzschild world since introducing a third twin moving on a radial timelike geodesic, first upwards and then downwards, one can show that his worldline is longer than those of the other twins. Usually in this and many other spacetimes two points may be connected by different geodesics and a worldline of the static observer and the multitude of possibilities concerning their lengths precludes generic conclusions such as that above.

Generically one can only seek for timelike curves having maximal length and this is the problem we are dealing with here. To the best of our knowledge the first who gave the correct but imprecise answer to the problem was Feynman (whilst at Princeton in the 1940): the longest worldline is a timelike geodesic. The answer may be deduced by analogy with the flat spacetime (geodesics are straight lines), but is insufficient if there are two or more timelike geodesics with common endpoints, as in the example above.

The general rigorous solution of the problem is achieved in two steps. The first step is contained in Hawking and Ellis' book [3] and we summarize it here in the form of three propositions.

Proposition 1. [1] *In any convex normal neighbourhood, if p and q can be joined by a timelike curve, then the unique timelike geodesic connecting them has length strictly greater than that of any other piecewise smooth timelike curve between the points.*

The existence of a convex normal neighbourhood is crucial here. If q does not lie in this neighbourhood of p then there are several timelike geodesics from p to q with different lengths, as in the Schwarzschild case outside the event horizon. But how to recognize whether given p and q can be connected by a *unique* timelike geodesic (i. e. that their neighbourhood is normal)? The first step described in [3] deals with bundles of nearby geodesics. Here the key notion is that of conjugate points.

Let $Z^{\mu}(s)$ be a geodesic deviation vector field on a timelike geodesic γ with tangent unit vector $u^{\alpha}(s)$. If Z^{μ} is chosen orthogonal to u_{μ}, then it satisfies the geodesic deviation equation

$$\frac{D^2}{ds^2} Z^{\mu} = R^{\mu}{}_{\alpha\beta\nu}\, u^{\alpha} u^{\beta} Z^{\nu}.$$

Any solution Z^{μ} of the equation is called a Jacobi field on γ. Points p and q on γ are said to be *conjugate* if there is Jacobi field $Z^{\mu} \neq 0$ such that $Z^{\mu}(q) = 0$ iff $Z^{\mu}(p) = 0$. If p and q are conjugate then one or more nearby geodesics intersect γ at p and q (or pass infinitesimally close to γ at these points) and they have different lengths. More precisely, if a geodesic γ joining points p_1 and p_2 has a point q conjugate to p_1 belonging to the segment $p_1 p_2$, then there exists a nearby timelike curve (not necessarily a geodesic) with endpoints p_1 and p_2 which is longer than γ. If there are no conjugate points, γ is the longest curve in the set of nearby curves.

Proposition 2. [2] *A timelike geodesic attains the local maximum of length (i. e. among nearby curves) from p_1 to p_2 iff there is no point conjugate to p_1 on the segment $p_1 p_2$.*

[1] Proposition 4.5.3 in [3]

[2] Proposition 4.5.8 in [3]

The very existence of conjugate points (but not their localization) is determined by

Proposition 3. [3] *If $R_{\alpha\beta}\, u^\alpha\, u^\beta \geq 0$ on a timelike geodesic γ and if the tidal force $R_{\mu\alpha\nu\beta}\, u^\alpha\, u^\beta \neq 0$ at some point p_0 on γ, there will be a pair of conjugate points somewhere on γ (providing that the geodesic can be extended sufficiently far).*

Returning to the twin paradox one concludes that the problem of which twin will be older at the reunion has a general (i. e. without computing the lengths of concrete worldlines) solution only if one of the twins' worldlines is a timelike geodesic free of conjugate points. From this conclusion one gets, e. g. that in Schwarzschild spacetime the radial geodesic (flight up and down), being free of conjugate points, is longer than the circular geodesic with the same endpoints since the latter has a conjugate point in the middle of its segment; this outcome is hard to derive from the analytic expressions for their lengths and and the two expressions can be compared only numerically.

This is, however, not the full solution of the problem we are interested in. Conjugate points allow one to find locally maximal curves but not the globally maximal ones. We are now entering the second step in solving the problem. The radial geodesic in Schwarzschild spacetime is longest in the bundle of nearby curves and is longer than the distant circular geodesic, but a priori there might exist a timelike geodesic, distant from the radial one and longer than that. We are looking for globally maximal length geodesics, i. e. curves whose length between their points attains maximal value. This maximal length is named *Lorentzian distance function* $d(p,q)$. Properties of maximal timelike geodesics, or curves realizing the distance function, are investigated in global Lorentzian geometry [4]. In [5] we quote six theorems from the book [4] which in our opinion are most relevant for searching globally maximal geodesics. Without glancing at them one may expect that these are mathematical 'theorems on existence' stating the presence of some global properties of geodesics if some global conditions hold. They cannot be and are not 'constructive'. In fact, in the search for locally maximal curves one investigates a bundle of close geodesics and their properties may be expressed in terms of the geodesic deviation equation. Yet in the problem of global maximality one compares the lengths of curves which besides their common endpoints are distant from each other. This means that there is no local analytic tool such as a differential equation (which expresses local properties of a mathematical object) allowing one to seek for the globally maximal curve. To find out which curve has the length equal to the distance function one has to study all timelike curves joining the given pair of points. In other terms there is no effective algorithmic procedure which provides in a *finite* number of steps the maximal (unique or not) timelike curve for the given endpoints. The search is not hopeless if one considers a spacetime with global symmetries (isometries). At least in the case of static spherically symmetric spacetimes it is possible to show that some class of timelike geodesics (the radial ones) consists of curves which are maximal on some segments.

[3] Proposition 4.4.2 in [3]

In this conference contribution we wish to make only a general introduction to the problem of maximal curves in Lorentzian spacetimes, which has evolved from the twin paradox in special relativity. Investigations of locally and globally maximal timelike curves are still at an initial stage and some results have been found in the simplest spacetimes of general relativity. These results reveal a multitude of possibilities concerning Jacobi fields and conjugate points. For details we refer the interested reader to author's recent work [6] and to his forthcoming papers.

References

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