On the Existence and Properties of Helically Symmetric Systems

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Abstract By an argument similar to that of Gibbons and Stewart, but in a different coordinate system and less restrictive gauge, we show that any weakly asymptotically simple, analytic vacuum or electro-vacuum spacetime which is periodic in time is necessarily stationary. We generalized this theorem to the presence of scalar fields and, among other results, derived new expressions for the Bondi mass in this case. Here we summarized these results and also briefly discuss some new considerations concerning the periodic solutions within linearized theory of gravity.

1 Introduction

The inspiral and coalescence of binary black holes or neutron stars appears to be the most promising source for the detectors of gravitational waves, so that there has been much effort going into the development of numerical codes and analytic approximation methods to find the corresponding solutions of Einstein's equations. One of the recent approaches [1, 2, 3], assumes the existence of a helical Killing vector k. In a co-rotating frame k generates time translations but it becomes null on the light cylinder and is spacelike outside. Hence, the spacetime is not stationary but it is still periodic in the region where k is spacelike. Helical symmetry implies equal

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amounts of outgoing and ingoing radiation present for all the time, so the spacetime is not expected to be asymptotically flat.

2 Non-existence of asymptotically flat periodic solutions periodic in time

In [4] the authors used the spin-coefficient formalism to prove that any asymptotically flat vacuum periodic solution of Einstein's equations is necessarily stationary. They have proved the existence of Killing vector ∂_u in the neighbourhood of null infinity $\mathscr{I}$ which, however, is null by construction everywhere and hence does not imply stationarity. In fact, even in the flat spacetime (stationary!) there is no Killing vector which is everywhere null and extends to a translation on $\mathscr{I}$. Therefore, the Minkowski spacetime is not stationary according to the definition given in [4]. In paper [5] we introduced a different coordinate system in which we were able to prove the existence of Killing vector K^a which is null on $\mathscr{I}$ but timelike in its neighbourhood, see figure 1. Moreover, in [5, 6] we generalized the proof for the presence of electromagnetic fields and scalar fields and derived new expressions for the Bondi mass of two kinds of scalar fields, massless Klein-Gordon and conformal scalar field[1]. Our main results are summarized in the following theorems and corollary.

Theorem 1. *A weakly asymptotically simple vacuum or electro-vacuum spacetime which is periodic in time and analytic in a neighbourhood of $\mathscr{I}$ necessarily has a Killing vector which is timelike in the interior and extends to a translation on $\mathscr{I}$. The same holds for spacetimes with massless Klein-Gordon fields[2] and for spacetimes with conformally invariant scalar fields.*

Corollary 1. *In any weakly asymptotically simple, stationary electro-vacuum spacetime which is analytic in a neighbourhood of $\mathscr{I}$, the electromagnetic field is also stationary. The same holds for spacetimes with massless Klein-Gordon fields.*

Theorem 2. *The Bondi mass of the spacetime which is a weakly asymptotically simple solution to Einstein-massless-Klein-Gordon equations is given in terms of the standard Newman-Penrose coefficients by*

$$M_B = -\frac{1}{2\sqrt{\pi}} \oint \left(\Psi_2^0 + \tfrac{1}{3}\,\partial_u(\phi^0\bar{\phi}^0) + \sigma^0\,\dot{\sigma}^0\right) \mathrm{d}S, \tag{1}$$

where ϕ is the complex scalar field. The corresponding mass-loss formula reads

$$\dot{M}_B = -\frac{1}{2\sqrt{\pi}} \oint \left(\dot{\sigma}^0\,\dot{\bar{\sigma}}^0 + 2\,\dot{\phi}^0\,\dot{\bar{\phi}}^0\right) \mathrm{d}S,$$

[1] Massless Klein-Gordon field is a solution to the standard wave equation $\nabla_a\nabla^a\phi = 0$ while the conformal scalar field satisfies the conformally invariant equation $(\nabla_a\nabla^a + R/6)\phi = 0$.

[2] In fact, this remains to be true also for a self-interacting scalar fields for which potential $V(\phi)$ is appropriately decaying at $\mathscr{I}$ (see [6]).

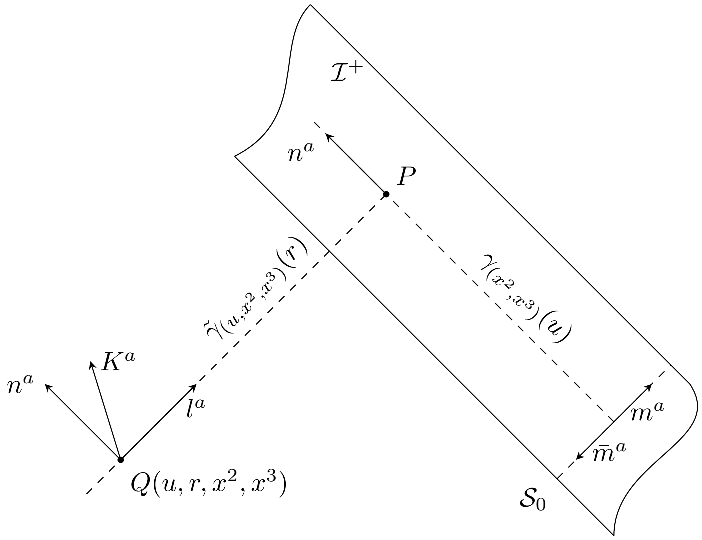

Fig. 1 The construction of the coordinate system and the Newman-Penrose null tetrad: the Killing vector K^a is null on $\mathscr{I}^+$ but timelike in its neighbourhood. Coordinate u is an affine parameter along null generators $\gamma_{(x^2,x^3)}(\mathrm{u})$ of $\mathscr{I}^+$, r is an affine parameter along null generators $\tilde{\gamma}_{(u,x^2,x^3)}(r)$ of hypersurfaces intersecting $\mathscr{I}$ at cuts $u =$ constant. On these two-sphere cuts, the coordinates x^2 and x^3 are introduced in an arbitrary way.

where the dot means the derivative with respect to time u.

The Bondi mass of the spacetime with conformal-scalar sources is given by the standard Newman-Penrose expression identical with (1) *for* $\phi^0 = 0$. *The Bondi mass-loss formula acquires the form*

$$\dot{M}_B = -\frac{1}{2\sqrt{\pi}} \oint \left(\dot{\sigma}^0 \dot{\bar{\sigma}}^0 + 2\,(\dot{\phi}^0)^2 - \phi^0\,\ddot{\phi}^0\right) \mathrm{d}S \tag{2}$$

where ϕ *is the real scalar field. Expression* (2) *is not negative semidefinite; a consequence of the null energy condition violation for conformally coupled scalar fields.*

3 Helical symmetry in linearized gravity

In this last section we turn to another aspect of the helical symmetry. Although no exact solutions of the Einstein equations possessing this symmetry are known, such solutions have been constructed in various "toy models" as, for example, in scalar gravity [7] or Nordström theory [8]. In [9] the authors have found solutions describing five dimensional asymptotically AdS black holes with scalar field. These spacetimes have only one Killing vector (and hence are not stationary and axisymmteric) of the form $K = \partial_t + \omega\partial_\psi$, which is tangent to the null generators of the horizon and can be asymptotically timelike, null or spacelike, depending on the parameters of the solution. Thus, these solutions exhibit a kind of helical symmetry.

Here we present some properties of helically symmetric solutions representing the fields of a particle moving on the circular orbit in the linearized Einstein's theory. We also show that it is feasible to achieve the equilibrium configuration of a binary system of such particles if both retarded and advanced solutions are taken into account.

First we find the field produced by a point particle of mass m_A ("particle A") moving uniformly along the circle of radius a with angular velocity ω in the plane $z=0$. We linearize the metric tensor in a usual way and introduce the trace-reversed perturbation $\bar{h}_{\mu\nu}$, subject to the de Donder gauge condition $\nabla^\mu \bar{h}_{\mu\nu} = 0$. Linearized Einstein's equations then acquire the well-known form

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}, \tag{3}$$

where $T_{\mu\nu}$ is an energy-momentum tensor corresponding to the point particle. The advanced $(-)$ and retarded $(+)$ solutions to the wave equation (3), after the transformation to the co-rotating frame, read

$$\begin{aligned}
\tilde{\bar{h}}^{\pm}_{00} &= -\frac{4m_A\gamma}{\rho_\pm}\left(1-\omega^2 ar\cos\theta_\pm\right)^2, & \tilde{\bar{h}}^{\pm}_{11} &= -\frac{4m_A\gamma}{\rho_\pm}\omega^2 a^2 \sin^2\theta_\pm,\\
\tilde{\bar{h}}^{\pm}_{01} &= -\frac{4m_A\gamma}{\rho_\pm}\left(1-\omega^2 ar\cos\theta_\pm\right)\omega a\sin\theta_\pm, & &\\
\tilde{\bar{h}}^{\pm}_{02} &= \frac{4m_A\gamma}{\rho_\pm}\left(1-\omega^2 ar\cos\theta_\pm\right)\omega ar\cos\theta_\pm, & &\\
\tilde{\bar{h}}^{\pm}_{12} &= \frac{4m_A\gamma}{\rho_\pm}\,\omega^2 a^2 r\sin\theta_\pm\cos\theta_\pm, & \tilde{\bar{h}}^{\pm}_{22} &= -\frac{4m_A\gamma}{\rho_\pm}\omega^2 a^2 r^2\cos^2\theta_\pm,
\end{aligned} \tag{4}$$

where the functions $\theta_\pm$ and $\rho_\pm$ are given implicitly by

$$\begin{aligned}
&\theta_\pm = \mp\omega R_\pm + \phi_0 - \hat{\phi}, \quad \rho_\pm = R_\pm \pm \omega ar\sin\theta_\pm,\\
&R_\pm = \sqrt{a^2+r^2+z^2-2ar\cos\theta_\pm}.
\end{aligned} \tag{5}$$

In terms of inertial coordinates, $\theta_\pm = \omega t_\pm + \phi_0 - \phi$, where $t_\pm = t \mp R_\pm$.

Due to the well-known inconsistency of the linearized theory, the solutions (4) do not obey the gauge condition imposed. However, we have $\nabla^\mu \bar{h}^\pm_{\mu\nu} = \mathscr{O}\left(\alpha^2\right)$, where $\alpha = \omega a/c$ (we set $c=1$), so our calculations are consistent up to order $\mathscr{O}(\alpha)$. In particular, the Ricci tensor does not vanish outside the world line of particle A, but $R_{\mu\nu} = \mathscr{O}\left(\alpha^2\right)$.

Gravitational field given by the advanced solution in (4) does not display usual peeling properties near future null infinity $\mathscr{I}^+$. In particular, the Newman-Penrose components of the Weyl spinor Ψ_m, $m = 0,\dots 4$, decay as $f(r)r^{-5+m}$, where $f(r)$ is an oscillating function of r.

The leading term in the asymptotic expansion of Ψ_4 for both advanced and retarded fields is

$$\Psi_4^{\pm} = \frac{m\alpha^4\gamma}{2r(1+\alpha\sin\theta_{\pm})}\left(10\alpha^2 + 15\alpha\sin\theta_{\pm} - \sin 3\theta_{\pm} + (2\alpha^2-8)\cos 2\theta_{\pm}\right), \quad (6)$$

where θ_+ is $r-$independent function given implicitly by (cf. (5))

$$\theta_+ = \alpha\cos\theta_+ + \alpha u$$

while θ_- is an oscillating function of r,

$$\theta_- = \omega(u+r) + \omega\sqrt{a^2+r^2-2ar\cos\theta_-}.$$

Thus, the retarded field decays in a usual way near $\mathscr{I}^+$ but the advanced field decays in an oscillatory manner in such a way that the limit of rescaled field does not exist at $\mathscr{I}^+$, see figure 2. Similar behaviour has been observed in the case of helically symmetric electromagnetic field, see [10].

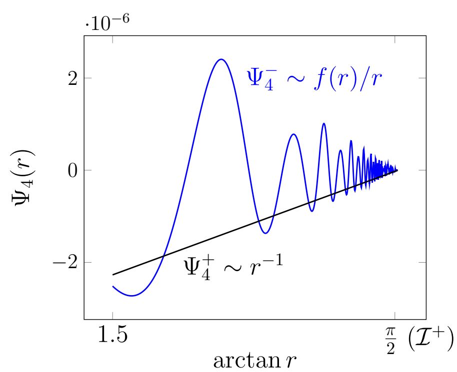

Fig. 2 The comparison of behaviour of Ψ_4 scalar in the neighbourhood of $\mathscr{I}^+$ in the case of the retarded solution (Ψ_4^+) and the advanced solution (Ψ_4^-). The value of $\Psi_4^{\pm}$ is plotted against rescaled coordinate $\arctan r$, so that the point $\pi/2$ on horizontal axis corresponds to $\mathscr{I}^+$. The retarded solution Ψ_4^+ exhibits usual r^{-1} decay while the advanced solution oscillates with increasing frequency.

Next we consider the motion of the test particle B in the spacetime with metric (4). For purely retarded (advanced) field of particle A, particle B is expected to move along the spiral with decreasing (increasing) radius as an inspection of retarded (advanced) effects suggests. This expectation is confirmed by the numerical solution of geodesic equation the results of which are shown in figure 3. In order to achieve a circular motion of particle B, it is necessary to take the solution in the time-symmetric form

$$\bar{h}_{\mu\nu} = \tfrac{1}{2}\left(\bar{h}^+_{\mu\nu} + \bar{h}^-_{\mu\nu}\right).$$

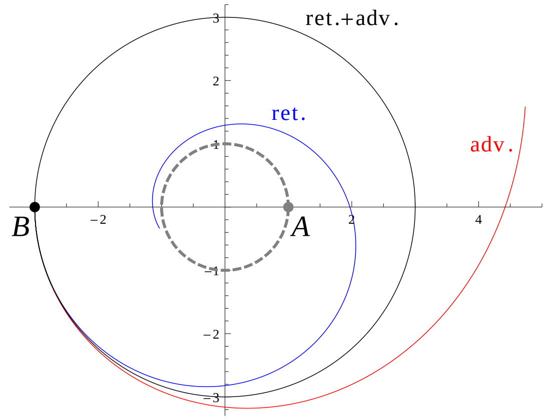

Fig. 3 The motion of test particle B in the field of particle A with mass $m = 0.44041$ (satisfying the equilibrium condition for the time-symmetric field) and with velocity $\alpha = 0.1$. The trajectories are plotted for purely retarded, purely advanced and for the time-symmetric field produced by particle A.

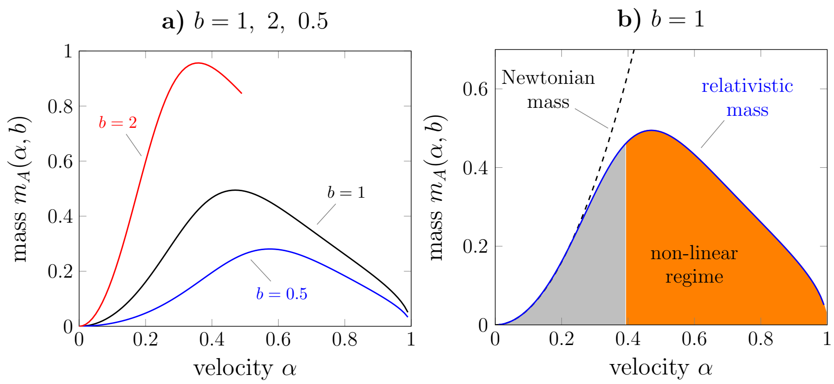

Fig. 4 **a)** The equilibrium value of the mass of particle A as a function of its velocity for selected values of radius b along which particle B orbits with velocity αb. **b)** Comparison with the Newtonian value of equilibrium mass for $b = 1$ when the circular orbits of both particles have the same radius. The orange region corresponds to the case when $|\bar{h}_{\mu\nu}| > |\eta_{\mu\nu}|$ so that the linearized theory breaks down.

With this choice, the angular acceleration $\ddot{\phi}$ becomes zero and the only remaining condition of equilibrium comes from the radial component of the geodesic equation, $\ddot{r} = 0$. For a given velocity, α, of particle A and for a given radius b of the orbit of particle B, the condition of equilibrium can be solved explicitly with respect to the mass m_A of particle A. In figure 4a) we plot the mass $m_A = m_A(\alpha, b)$ as a function of the velocity α for selected values of parameter b (we use units in which $a = 1$); in 4b) the relativistic results are compared with the Newtonian results for $b = 1$.

Thus, if the mass $m_A(\alpha, b)$ is chosen so as to satisfy the equilibrium condition, $\ddot{r} = 0$, particle B will move along the circle of radius b with the velocity αb in the (time-symmetric) field of particle B.

A complete discussion of the equilibrium of a binary system on circular orbits will be given in [11].

Acknowledgement

This work was supported by the grant GAČR 202/09/00772 of the Czech Republic.

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