Hamiltonian formalism for spinning black holes in general relativity

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Abstract A Hamiltonian treatment of gravitationally interacting spinning black holes is presented based on a tetrad generalization of the Arnowitt-Deser-Misner (ADM) canonical formalism of general relativity. The formalism is valid through linear order in the single spins. For binary systems, higher-order post-Newtonian Hamiltonians are given in explicit analytic forms. A next-to-leading order in spin generalization is presented, others are mentioned. Comparisons between the Hamiltonian formalisms by ADM, Dirac, and Schwinger are made.

1 Introduction

About half a century after Einstein's invention of general relativity, Hamiltonian formulations of the theory became available. Dirac developed his formalism in the years 1958 - 1959 [1, 2, 3], Arnowitt, Deser, and Misner (ADM) in the years 1959 - 1960 [4, 5, 6], for a summary see [7], and Schwinger in 1963 [8]. The three formalisms came up from quite different action functionals: Dirac used the Einstein action, ADM the Einstein-Hilbert action in Palatini form, and Schwinger the tetrad-generalized Palatini-based Einstein-Hilbert action. In the following, the three approaches are summarized and compared. Often in the article, the units $16\pi G = 1$ and $c = 1$ are applied.

In the context of his constraint dynamics formalism with various typs of constraints, weakly and strongly vanishing ones, Dirac gave the full Hamiltonian, i.e. before applying the constraint equations and coordinate conditions, in the form, with ∂_i denoting partial space-coordinate derivative and i^0 spatial infinity,

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$$H_{\mathrm{D}} = -\int d^3x \partial_i (g^{-1/2} \partial_j (g\gamma^{ij})) + \int d^3x (N\mathscr{H} - N^i \mathscr{H}_i)$$
$$= -\oint_{i^0} dS_i g^{-1/2} \partial_j (g\gamma^{ij}) + \int d^3x (N\mathscr{H} - N^i \mathscr{H}_i) \tag{1}$$

(correctness of this form within the full non-linear theory was shown only in 1974, by Regge and Teitelboim [9]), whereas ADM and Schwinger gave the Hamiltonian in fully reduced forms, i.e. after applying constraint equations and appropriate coordinate conditions,

$$H_{\mathrm{ADM}} = \int d^3x \partial_i \partial_j (g_{ij} - \delta_{ij} g_{kk}) = \oint_{i^0} dS_i \partial_j (g_{ij} - \delta_{ij} g_{kk}), \tag{2}$$

$$H_{\mathrm{S}} = -\int d^3x \partial_i \partial_j (g\gamma^{ij}) = -\oint_{i^0} dS_i \partial_j (g\gamma^{ij}). \tag{3}$$

Though derived under specific coordinate conditions, here the form of the three-dimensional metric g_{ij}, $(i, j = 1, 2, 3)$, with inverse metric γ^{ij}, is still left general to better illuminate the general expressions behind. All three surface integrals coincide under the assumptions made: asymptotic flat spacetimes with coordinates of the form $N = 1 + O(1/r)$, $g_{ij} = \delta_{ij} + O(1/r)$, and $N^i = O(1/r)$ at spacelike infinity $(r \to \infty)$ including the first derivatives of the metric functions to decay as $1/r^2$. The Lagrangian multipliers $N \equiv (-g^{00})^{-1/2}$ and $N^i \equiv \gamma^{ij} g_{0i}$ are respectively coined "lapse" and "shift" functions by Wheeler [10]. The clear identification of H_{ADM} with the total energy of the system is one of the merits of ADM in the Hamiltonian approach to general relativity. The constraint equations read,

$$\mathscr{H} = 0 \quad \text{and} \quad \mathscr{H}_i = 0, \tag{4}$$

with the Hamilton density of weight one, used by ADM and Dirac,

$$\mathscr{H} \equiv -g^{1/2}\mathrm{R} + \frac{1}{g^{1/2}} \left(g_{ik} g_{jl} \pi^{ij} \pi^{kl} - \frac{1}{2} (g_{ij} \pi^{ij})^2 \right) + \mathscr{H}_M \quad \text{(ADM)}$$
$$= B + \partial_i (g^{-1/2} \partial_j (g\gamma^{ij})) + \frac{1}{g^{1/2}} \left(g_{ik} g_{jl} \pi^{ij} \pi^{kl} - \frac{1}{2} (g_{ij} \pi^{ij})^2 \right) + \mathscr{H}_M \quad \text{(D)}, \tag{5}$$

where

$$B \equiv \frac{1}{4} g^{1/2} g_{ij,k} g_{lm,n} [(\gamma^{il}\gamma^{jm} - \gamma^{ij}\gamma^{lm})\gamma^{kn} + 2(\gamma^{ik}\gamma^{lm} - \gamma^{il}\gamma^{mk})\gamma^{jn}], \tag{6}$$

or with Schwinger's weight-two density,

$$g^{1/2}\mathscr{H} = Q + \partial_i \partial_j q^{ij} + q^{ik} q^{jl} \Pi_{ij} \Pi_{kl} - (q^{ij} \Pi_{ij})^2 + g^{1/2} \mathscr{H}_M \quad \text{(S)}, \tag{7}$$

where

$$Q \equiv -\frac{1}{4} q^{mn} \partial_m q^{kl} \partial_n q^{kl} - \frac{1}{2} q_{ln} \partial_m q^{kl} \partial_k q^{mn} - \frac{1}{2} q^{kl} \partial_k \ln(q^{1/2}) \partial_l \ln(q^{1/2}), \tag{8}$$

with $\Pi_{ij} = -g^{-1}(\pi_{ij} - \frac{1}{2}\pi g_{ij})$ and $q^{ij} = g\gamma^{ij}$, $q = g^2$, where g denotes the determinant of g_{ij} and lowering of indices is with g_{ij}, $\pi = g_{ij}\pi^{ij}$. The Dirac and ADM canonical field momentum is given by $\pi_{ij} = -g^{1/2}(K_{ij} - Kg_{ij})$, with $K = \gamma^{ij}K_{ij}$, where $K_{ij} = -N\Gamma^0_{ij}$ is the extrinsic curvature of a spacelike hypersurface defined through constant-in-time slice, $t = x^0 =$ const. Γ^0_{ij} denote Christoffel symbols. Schwinger's canonical field momentum Π_{ij} is just $g^{-1/2}K_{ij}$. The intrinsic curvature scalar reads R. The expressions $g_{ij}dx^i dx^j$ and $K_{ij}dx^i dx^j$ are respectively called first and second fundamental form of a hypersurface. Both tensors g_{ij} and K_{ij} are symmetric.

The momentum density of weight one takes the forms

$$\begin{aligned} \mathscr{H}_i &\equiv 2g_{ij}D_k\pi^{jk} + \mathscr{H}_{Mi} \quad \text{(ADM and D)} \\ &= -\Pi_{lm}\partial_i q^{lm} + \partial_i(2\Pi_{lm}q^{lm}) - \partial_l(2\Pi_{im}q^{lm}) + \mathscr{H}_{Mi} \quad \text{(S)}, \end{aligned} \tag{9}$$

where D_i denotes the three-dimensional covariant space derivative. The given densities are densities with respect to three-dimensional coordinate transformations. $\mathscr{H}_M$ and $\mathscr{H}_{Mi}$ are matter densities.

In 1967, DeWitt [11] and more refined later on in 1974, Regge and Teitelboim [9] explicitly showed that in asymptotically flat spacetimes the Hamiltonian, before applying constraint equations and coordinate conditions, takes the form

$$H = \oint_{i^0} dS_i \partial_j(g_{ij} - \delta_{ij}g_{kk}) + \int d^3x(N\mathscr{H} - N^i\mathscr{H}_i). \tag{10}$$

This general Hamiltonian - not yet to be identified with the energy of the system -, delivers all field equations also including those for the lapse and shift functions after imposing appropriate coordinate conditions.

Undoubtedly, originating from the very useful coordinate choice made by ADM - one may call it "maximal isotropic" -,

$$\pi^{ii} = 0, \quad 3\partial_j g_{ij} - \partial_i g_{jj} = 0 \quad \text{or} \quad g_{ij} = \psi\delta_{ij} + h^{\mathrm{TT}}_{ij}, \tag{11}$$

the ADM formalism became the most often applied canonical formalism. The independent field variables therein are π^{ij}_{TT} and h^{TT}_{ij}. Both variables are traceless and tracefree (TT). Already in 1961 Kimura [12] used this formalism for applications. The Poisson bracket reads

$$\{F, G\} = \delta^{\mathrm{TT}kl}_{ij}\left(\frac{\delta F}{\delta h^{\mathrm{TT}}_{ij}}\frac{\delta G}{\delta \pi^{kl}_{\mathrm{TT}}} - \frac{\delta G}{\delta h^{\mathrm{TT}}_{ij}}\frac{\delta F}{\delta \pi^{kl}_{\mathrm{TT}}}\right), \tag{12}$$

with

$$\delta^{\mathrm{TT}kl}_{ij} = \frac{1}{2}(P_{il}P_{jk} + P_{ik}P_{jl} - P_{kl}P_{ij}), \quad P_{ij} = \delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}, \tag{13}$$

where $1/\nabla^2$ denotes the inverse Laplacian. The nonlocality of the TT-operator $\delta^{\mathrm{TT}kl}_{ij}$ is just the gravitational analogue of the well-known nonlocality of the Coulomb gauge in the electrodynamics. If Schwinger would have chosen coordinate conditions corresponding to those introduced above (ADM also introduced another set of coordinate conditions to which Schwinger adjusted),

$$\Pi_{ii} = 0, \quad q^{ij} = \varphi\delta_{ij} + f^{ij}_{\mathrm{TT}}, \tag{14}$$

a similar simple technical formalism for detailed calculations would have resulted with the independent field variables Π^{TT}_{ij} and f^{ij}_{TT}. To our best knowledge, only the paper by Kibble [13] delivers an application of Schwinger's formalism, apart from Schwinger himself, namely a Hamilton formulation of the Dirac spinor field in gravity. Much later in 1978, Nelson and Teitelboim [14] completed the same task within the tetrad-generalized Dirac formalism [15]. The Poisson bracket in the Schwinger formalism resembles very much the ADM one.

In terms of the ADM variables, Schwinger's fundamental field components take the quite simple form, [16],

$$q^{ij} = (\psi^2 - \frac{1}{2}h^{\mathrm{TT}}_{kl}h^{\mathrm{TT}}_{kl})\delta_{ij} - \psi h^{\mathrm{TT}}_{ij} + h^{\mathrm{TT}}_{ik}h^{\mathrm{TT}}_{kj}. \tag{15}$$

Dirac on the other side had chosen the following coordinate system or gauge, called "maximal slicing" because of the field momentum condition,

$$\pi = 0, \quad \partial_j(g^{1/3}g^{ij}) = 0. \tag{16}$$

The corresponding independent field variables are

$$\tilde{\pi}^{ij} = (\pi^{ij} - \frac{1}{3}\gamma^{ij}\pi)g^{1/3}, \quad \tilde{g}_{ij} = g^{-1/3}g_{ij}. \tag{17}$$

To leading order linear in the metric functions, the Dirac gauge coincides with the ADM gauge. The full reduction of the Dirac-form dynamics to the independent degrees of freedom has been performed by Regge and Teitelboim [9]. The Poisson bracket reads

$$\{F,G\} = \tilde{\delta}^{kl}_{ij}\left(\frac{\delta F}{\delta\tilde{g}_{ij}}\frac{\delta G}{\delta\tilde{\pi}^{kl}} - \frac{\delta G}{\delta\tilde{g}_{ij}}\frac{\delta F}{\delta\tilde{\pi}^{kl}}\right) + \frac{1}{3}(\tilde{\pi}^{ij}\tilde{g}^{kl} - \tilde{\pi}^{kl}\tilde{g}^{ij})\frac{\delta F}{\delta\tilde{\pi}^{ij}}\frac{\delta G}{\delta\tilde{\pi}^{kl}}, \tag{18}$$

with

$$\tilde{\delta}^{kl}_{ij} = \frac{1}{2}(\delta^k_i\delta^l_j + \delta^l_i\delta^k_j) - \frac{1}{3}\tilde{g}_{ij}\tilde{g}^{kl}, \quad \tilde{g}_{ij}\tilde{g}^{jl} = \delta^l_i. \tag{19}$$

The following quoted results, apart from the last one, are based on the ADM approach. The first post-Newtonian (1PN) equations of motion, usually called Einstein-Infeld-Hoffmann equations of motion, were rederived by Kimura in 1961 [12]. In 1974, Ohta et al. [17] derived the 2PN binary equations with shortcomings corrected by Damour and Schäfer [18] only much later. About the same time, in 1985, Schäfer succeeded with the dissipative 2.5PN level, [16]. Using for the first time in post-Newtonian calculations the technique of dimensional regularization, the 3PN binary Hamiltonian was obtained in 2001 by Damour, Jaranowski, and Schäfer

[19]. Much simpler to derive was the dissipative 3.5PN level [20]. Based on a post-linear paper by Schäfer [21], the first post-Minkowskian (1PM) n-body Hamiltonian was achieved in closed form by Ledvinka, Schäfer, and Bičák in 2008 [22]. The general relativistic Hamilton dynamics of compact objects with spin (proper rotation) has found several explicit results. Counting the order of the spin as $1/c$ (notice the spin of a miximally rotating black hole reading GM^2/c), the leading order spin-orbit coupling is decribed by an 1.5PN Hamiltonian [23], the next to leading order one by a 2.5PN Hamiltonian, [24, 25], and the next-to-next to leading order one by a 3.5PN Hamiltonain [26]. The leading order radiation damping from spin-orbit coupling is decribed by an 4PN Hamiltonian [27]. In case of spin(1)-spin(2) coupling, the conservative 2PN Hamiltonian is given in [23], the 3PN Hamiltonian by [28, 25], and the 4PN one by [29]. The 4.5PN Hamiltonian of the leading order radiation damping from spin(1)-spin(2) coupling is given in [30]. Results on the spin(1)-spin(1) coupling are the 2PN and 3PN Hamiltonians for black holes by respectively [23] and [31]. Several leading higher-order-in-spin Hamiltonians were obtained from Kerr metric considerations, [32]. Spinning test-particles in the Kerr metric have been treated by Barausse, Racine, and Buonanno in 2009, [33] using Dirac's constraint dynamics formalism.

2 Analytic representation of binary black holes – the Brill-Lindquist initial value solution

The model used in this article to describe compact objects are Dirac delta functions. In this section it will be shown that Dirac delta functions can indeed be used to describe black holes in interaction with each other.

An isolated black hole with mass m is represented through the line element

$$\begin{aligned} ds^2 &= -\left(\frac{1-\frac{Gm}{2rc^2}}{1+\frac{Gm}{2rc^2}}\right)^2 c^2dt^2 + \left(1+\frac{Gm}{2rc^2}\right)^4 \delta_{ij}dx^i dx^j \\ &= -\left(\frac{1-\frac{Gm}{2Rc^2}}{1+\frac{Gm}{2Rc^2}}\right)^2 c^2dt^2 + \left(1+\frac{Gm}{2Rc^2}\right)^4 \delta_{ij}dX^i dX^j \end{aligned} \quad (20)$$

when isotropic coordinates x^i with $r^2 = x^i x^i$ and X^i with $R^2 = X^i X^i$ are employed. The two coordinate systems are related through the inversion map $R = \left(\frac{Gm}{2c^2}\right)^2 / r$. Obviously, the line element shows isometry under this map. For binary black holes, initially at rest, the metric at that instant of time may have the form

$$ds^2 = -\left(\frac{1-\frac{\beta_1 G}{2r_1c^2}-\frac{\beta_2 G}{2r_2c^2}}{1+\frac{\alpha_1 G}{2r_1c^2}+\frac{\alpha_2 G}{2r_2c^2}}\right)^2 c^2dt^2 + \left(1+\frac{\alpha_1 G}{2r_1c^2}+\frac{\alpha_2 G}{2r_2c^2}\right)^4 d\mathbf{x}^2 \quad (21)$$

as shown by Brill and Lindquist in 1963, [34], for the space part of the given metric. The time part of the given metric has been derived later by Jaranowski and Schäfer [35]. The coefficients α_a and β_a, $a=1,2$ do depend on the masses and the relative coordinate distance of the two black holes $r_{12}^2=(x_1^i-x_2^i)(x_1^i-x_2^i)$. Furthermore, $r_a^2=(x^i-x_a^i)(x^i-x_a^i)$, where x_a^i are the position vectors of the black holes. Notice, our variables are living in an euclidean space x^i which is conformally related with the physical one. The geometrical interpretation of Brill-Lindquist black holes are two Einstein-Rosen bridges, both starting in the same physical space but ending in two different other ones. The energy of the Brill-Lindquist black hole configuration reads

$$E_{ADM}=-\frac{c^4}{2\pi G}\oint_{i_0} ds_i\partial_i\Psi=(\alpha_1+\alpha_2)c^2, \tag{22}$$

where $\Psi=1+\frac{\alpha_1 G}{2r_1c^2}+\frac{\alpha_2 G}{2r_2c^2}$, where $\Delta\Psi=0$ for $x^i\neq x_1^i,x_2^i$ with $\Delta\equiv\nabla^2$. The inversion map of the three-metric of black hole, say 1, at its throat reads $x_1'^i=x_1^i\alpha_1^2G^2/4c^4r_1^2$, where $x_1'^i=x'^i-x_1^i,\quad x_1^i=x^i-x_1^i,\quad r_1=|x^i-x_1^i|$. The three-metric line element dl^2 takes the forms

$$\begin{aligned} dl^2=\Psi^4d\mathbf{x}^2&=\left(1+\frac{\alpha_1G}{2r_1c^2}+\frac{\alpha_2G}{2r_2c^2}\right)^4d\mathbf{x}^2\\ &=\Psi'^4d\mathbf{x}'^2=\left(1+\frac{\alpha_1G}{2r_1'c^2}\left(1+\frac{\alpha_2G}{2r_2c^2}\right)\right)^4d\mathbf{x}'^2, \end{aligned} \tag{23}$$

with $\Psi'=1+\frac{\alpha_1G}{2r_1'c^2}+\frac{\alpha_1\alpha_2G^2}{4r_2r_1'c^4}$ and $r_2^i=\frac{\alpha_1^2G^2}{4c^4}\frac{r_1'^i}{r_1'^2}+r_{12}^i,\quad r_{12}^i=r_1^i-r_2^i=x_2^i-x_1^i$. Hereof, by definition, the rest-mass of black hole 1 comes out to read

$$m_1\equiv-\frac{c^2}{2\pi G}\oint_{i_0}ds_i'\partial_i'\Psi'=\alpha_1+\frac{\alpha_1\alpha_2G}{2r_{12}c^2}. \tag{24}$$

The calculations presented above treat the metric functions geometrically as pure vacuum solution without sources. No divergences occur. Infinities only mean infinite distances. In the following it will be shown how to reconstruct the Brill-Lindquist initial value solution with the aid of Dirac delta functions.

Let us have a look at the constraint equations with point-mass sources,

$$g^{1/2}\mathrm{R}-\frac{1}{g^{1/2}}\left(\pi_j^i\pi_i^j-\frac{1}{2}\pi_i^i\pi_j^j\right)=\frac{16\pi G}{c^3}\sum_a\left(m_a^2c^2+\gamma^{ij}p_{ai}p_{aj}\right)^{1/2}\delta_a \tag{25}$$

(identical with $2\sqrt{-g}G^{00}=\frac{16\pi G}{c^4}\sqrt{-g}T^{00}$ in standard notation),

$$-2\partial_j\pi_i^j+\pi^{kl}\partial_ig_{kl}=\frac{16\pi G}{c^3}\sum_a p_{ai}\delta_a \tag{26}$$

($2\sqrt{-g}G_i^0 = \frac{16\pi G}{c^4}\sqrt{-g}T_i^0$). δ_a is an abbreviation of $\delta(x^i - x_a^i)$ with x_a^i the position vector of mass a ($\int d^3x\delta_a = 1$). Using the ADM gauge in the form

$$g_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\rm TT}, \quad [3\partial_j g_{ij} - \partial_i g_{jj} = 0],$$
$$\pi^{ii} = 0, \quad \text{or} \quad \pi^{ij} = \partial_i\pi^j + \partial_j\pi^i - \frac{2}{3}\delta_{ij}\partial_k\pi^k + \pi_{\rm TT}^{ij}, \tag{27}$$

the constraint equations simplify to the following equation, imposing $h_{ij}^{\rm TT} = \pi_{\rm TT}^{ij} = p_{ai} = 0$,

$$-\left(1 + \frac{1}{8}\,\phi\right)\Delta\phi = \frac{16\pi G}{c^2}\sum_a m_a\delta_a. \tag{28}$$

With the aid of the ansatz

$$\phi = \frac{4G}{c^2}\left(\frac{\alpha_1}{r_1} + \frac{\alpha_2}{r_2}\right), \tag{29}$$

after Hadamard partie finie regularization, the constraint equation yields,

$$m_a = \alpha_a + \frac{\alpha_a\alpha_b G}{2r_{ab}c^2}, \quad a \neq b, \tag{30}$$

$$\alpha_a = m_a - \frac{m_a + m_b}{2} + \frac{c^2 r_{ab}}{G}\left(\sqrt{1 + \frac{m_a + m_b}{c^2 r_{ab}/G} + \left(\frac{m_a - m_b}{2c^2 r_{ab}/G}\right)^2} - 1\right). \tag{31}$$

The Hamiltonian clearly results in

$$H_{\rm BL} = (\alpha_1 + \alpha_2)\,c^2 = (m_1 + m_2)\,c^2 - G\,\frac{\alpha_1\alpha_2}{r_{12}}. \tag{32}$$

The d-dimensional generalization of the above treatment runs as follows. The d-metric functions read,

$$g_{ij} = \psi^{\frac{4}{d-2}}\delta_{ij}, \quad \Psi = 1 + \frac{1}{4}\frac{d-2}{d-1}\phi,$$
$$\phi = \frac{4G}{c^2}\frac{\Gamma(\frac{d-2}{2})}{\pi^{\frac{d-2}{2}}}\left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}}\right). \tag{33}$$

The d-dimensional inverse Laplacian Δ^{-1} takes the form,

$$-\Delta^{-1}\delta_a = \frac{\Gamma((d-2)/2)}{4\pi^{d/2}}r_a^{2-d}, \tag{34}$$

where Γ denotes the Euler gamma function. The ansatz for Ψ thus reads

$$\Psi = 1 + \frac{G(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}} \right) \tag{35}$$

and the constraint equation takes the form

$$\left(1 + \frac{G(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}} \right) \right) \alpha_a \delta_a = m_a \delta_a. \tag{36}$$

Chosing $1 < d < 2$, a fully finite result comes in the form,

$$\left(1 + \frac{G(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \frac{\alpha_b}{r_{12}^{d-2}} \right) \alpha_a \delta_a = m_a \delta_a, \quad a \neq b. \tag{37}$$

Analytic continuation through $d = 3$ can now be performed without facing divergences.

3 Spin in Minkowski space

Before entering into gravity in asymptotically flat spacetimes, a discussion of spin in Minkowski spacetime seems most convenient. Using canonical variables, the total angular momentum $\mathbf{J} = (J^i) = (J_i)$ of a particle with spin vector $\hat{\mathbf{S}}$ reads

$$\mathbf{J} = \hat{\mathbf{X}} \times \mathbf{P} + \hat{\mathbf{S}}, \tag{38}$$

where $\hat{\mathbf{X}}$ denotes the canonical position vector and $\mathbf{P}$ its canonical linear momentum. The Lorentz boost is given by

$$\mathbf{K} \equiv -\mathrm{t}\mathbf{P} + \mathbf{G} = -\mathrm{t}\mathbf{P} + \mathrm{H}\hat{\mathbf{X}} - \frac{1}{H+m} \hat{\mathbf{S}} \times \mathbf{P} \tag{39}$$

with the center-of-energy vector $\mathbf{G}$. The free-particle Hamiltonian $H = \sqrt{m^2 + \mathbf{P}^2}$.

The center-of-energy position vector is given by

$$\bar{\mathbf{X}} = \hat{\mathbf{X}} - \frac{1}{(H+m)H} \hat{\mathbf{S}} \times \mathbf{P}, \tag{40}$$

thus, $\mathbf{G} = H\bar{\mathbf{X}}$. The center-of-spin vector, or Newton-Wigner position vector, is defined by $\hat{\mathbf{X}}$. The Poisson brackets of its components vanish $\{\hat{X}^i, \hat{X}^j\} = 0$. The center-of-energy position vector $\bar{\mathbf{X}} = \hat{\mathbf{X}} - \frac{1}{(H+m)H} \hat{\mathbf{S}} \times \mathbf{P}$ has non-vanishing Poisson brackets of its components and the center-of-inertia position vector $\mathbf{X} = \hat{\mathbf{X}} + \frac{1}{(H+m)m} \hat{\mathbf{S}} \times \mathbf{P}$ as well. Using the center-of-inertia position vector, in four-dimensional language, $S^{\mu\nu}P_\nu = 0$ holds, for the center-of-energy position vector $\bar{S}^{\mu\nu}n_\nu = 0$, $n_\mu = (-1,0,0,0)$ is valid, and for the center-of-spin position vector $m\hat{S}^{\mu\nu}n_\nu + \hat{S}^{\mu\nu}P_\nu = 0$ happens. The reason for those different spin-supplementary conditions is rooted in the invariance of the total angular momentum against shift of coordinates.

For isolated systems, the Poincaré algebra is valid,

$$\begin{aligned}
\{P_i,\mathrm{H}\} = \{J_i,H\} &= 0, \qquad \{J_i,P_j\} = \varepsilon_{ijk}P_k, \\
\{J_i,J_j\} = \varepsilon_{ijk}J_k, \qquad \{J_i,G_j\} &= \varepsilon_{ijk}G_k, \qquad \{G_i,H\} = P_i, \\
\{G_i,P_j\} = \frac{1}{c^2}H\,\delta_{ij}, \qquad \{G_i,G_j\} &= -\frac{1}{c^2}\varepsilon_{ijk}J_k.
\end{aligned} \tag{41}$$

G_i is not a constant of motion, but K_i is,

$$d\mathbf{K}/dt = \partial\mathbf{K}/\partial t + \{\mathbf{K},H\} = -\mathbf{P} + \{\mathbf{G},H\} = 0. \tag{42}$$

For many-particle systems with interaction, it generally holds,

$$\mathbf{P} = \sum_a \mathbf{p}_a, \qquad \mathbf{J} = \sum_a (\mathbf{r}_a \times \mathbf{p}_a + \mathbf{s}_a), \tag{43}$$

$$M \equiv \sqrt{H^2 - \mathbf{P}^2}, \qquad H = \sqrt{M^2 + \mathbf{P}^2}, \tag{44}$$

$$\hat{\mathbf{X}} = \frac{\mathbf{G}}{H} + \frac{1}{M(H+M)}(\mathbf{J} - \frac{\mathbf{G}}{H} \times \mathbf{P}) \times \mathbf{P}, \tag{45}$$

with

$$\begin{aligned}
\{\hat{X}^i,\hat{X}^j\} = \{P^i,P^j\} &= 0, \qquad \{\hat{X}^i,P^j\} = \delta^{ij}, \\
\{M,\hat{X}^j\} = \{M,P^j\} &= \{M,H\} = 0.
\end{aligned} \tag{46}$$

For free particles with spin, one has, $H = \sum_a h_a$, with $h_a = \sqrt{m_a^2 + \mathbf{p}_a^2}$, and $\mathbf{G} = \sum_a (h_a \mathbf{r}_a - \frac{1}{h_a + m_a}\mathbf{s}_a \times \mathbf{p}_a)$.

4 Spin and gravity – asymptotic flat spacetimes

The treatment of spin in gravity is most conveniently achieved by the introduction of a tetrad field e_a^μ ($\mu = 0,1,2,3; a = 0,1,2,3$) having the properties $e_a^\mu e_{b\mu} = \eta_{ab}$ and $e_{a\mu}e_{b\nu}\eta^{ab} = g_{\mu\nu} = g_{\nu\mu}$. Local Lorentz transformations are defined by $e_a'^\mu = L^b{}_a e_b^\mu$ with $L^a{}_c \eta_{ab} L^b{}_d = \eta_{cd}$. The condition of homogeneous transformation of the space-time derivative of a physical object ϕ under local Lorentz transformations introduces a linear connection ω_μ^{ab},

$$D_\mu \phi \equiv \partial_\mu \phi + \frac{1}{2}\omega_\mu^{ab} G_{[ab]}\phi, \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \tag{47}$$

with transformation property $\omega'^{ab}_\mu = L^a{}_c L^b{}_d \omega_\mu^{cd} + L^a{}_d \partial_\mu L^{bd}$. The object $G_{[ab]}$ is defined through infinitesimal local Lorentz transformations of ϕ, $\delta\phi = -\delta\xi^{[ab]} G_{[ab]}\phi$, with infinitesimal group parameters $\delta\xi^{[ab]}$.

The curvature tensor $R^{ab}_{\mu\nu}$ is defined by

$$D_\mu D_\nu \phi - D_\nu D_\mu \phi = R^{ab}_{\mu\nu} G_{[ab]}\phi, \tag{48}$$

$$R^{ab}_{\mu\nu} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\nu^{ac}\omega_\mu^{bd}\eta_{cd} - \omega_\mu^{ac}\omega_\nu^{bd}\eta_{cd}. \tag{49}$$

The simplest Lagrangian density (apart from the herein trivial cosmological constant) for the gravitational field reads,

$$\mathscr{L}_G = \det(e^c_\gamma) e^\mu_a e^\nu_b R^{ab}_{\mu\nu}(\omega) + \partial_\mu \mathscr{C}^\mu, \tag{50}$$

where the exact divergence $\partial_\mu \mathscr{C}^\mu$ is needed to render the variational principle valid for variations with prescribed properties at the boundary of the four-dimensional integration area,

The vacuum field equations read

$$0 = \frac{\delta \mathscr{L}_G}{\delta e^\mu_a} = \det(e^c_\gamma)(2R^{ab}_{\mu\sigma} e^\sigma_b - e^a_\mu R^{cb}_{\rho\sigma} e^\rho_c e^\sigma_b), \tag{51}$$

$$0 = \frac{\delta \mathscr{L}_G}{\delta \omega_\mu^{ab}} \quad \Rightarrow \quad \omega_\mu^{ab} = \omega_\mu^{ab}(e, \partial_\nu e). \tag{52}$$

The latter equation reduces the gravitational field to the Einsteinian one with suppressed torsion. This property will be kept in the following when treating spinning sources of the gravitational field.

The matter action $W_M = \int d^4x[\mathscr{L}_M + \mathscr{L}_C]$ for a spinning classical object can be put into the form, e.g. [36], where the Lagrangian density of the dynamical part reads,

$$\mathscr{L}_M = \int d\tau \left[\left(p_\mu - \frac{1}{2} S_{ab}\,\omega_\mu^{ab} \right) \frac{dz^\mu}{d\tau} + \frac{1}{2} S_{ab} \frac{d\theta^{ab}}{d\tau} \right] \delta_{(4)} \tag{53}$$

and where the constraints part is given by

$$\mathscr{L}_C = \int d\tau \left[\lambda_1^a p^b S_{ab} + \lambda_{2[i]} \Lambda^{[i]a} p_a - \frac{\lambda_3}{2}(p^2 + m^2) \right] \delta_{(4)}. \tag{54}$$

$\delta_{(4)}$ is a four-dimensional Dirac delta function, $\delta_{(4)} = \delta(x^\mu - z^\mu)$, ($\int d^4x \delta_{(4)} = 1$), τ is a proper time variable, p_μ and z^μ denote respectively the four-dimensional kinetic momentum form and position vector of the particle and S_{ab} is its spin tensor. The angle variables θ^{ab} with $d\theta^{ab} = \Lambda_C{}^a d\Lambda^{Cb} = -d\theta^{ba}$ are anholonomic ones. Capital indices refer to body-fixed Lorentz-frame coordinates. λ_1, λ_2, and λ_3 are Lagrangian multipliers.

The equations of motion resulting hereof reads,

$$\frac{DS_{ab}}{D\tau} = 0, \tag{55}$$

$$\frac{Dp_\mu}{D\tau} = -\frac{1}{2}\mathrm{R}_{\mu\rho ab}u^\rho S^{ab}, \qquad u^\mu \equiv \frac{dz^\mu}{d\tau} = \lambda_3 p^\mu. \tag{56}$$

They completely correspond to the divergence freeness or dynamics of the Tulczyjew stress-energy tensor for pole-dipole particles,

$$N\sqrt{g}T^{\mu\nu} = \int d\tau \left[\lambda_3 p^\mu p^\nu \delta_{(4)} + \left(u^{(\mu} S^{\nu)\alpha}\delta_{(4)}\right)_{||\alpha}\right]. \tag{57}$$

The index symbol $||$ denotes covariant four-dimensional derivative.

4.1 Hamiltonian for self-gravitating spinning compact objects

Variation of the matter action with respect to the Lagrangian multipliers λ_1, λ_2, and λ_3 results in the relations, respectively,

$$nS_i \equiv n^\mu S_{\mu i} = \frac{p_k \gamma^{kj} S_{ji}}{np} = g_{ij} n S^j, \tag{58}$$

$$\Lambda^{[j](0)} = \Lambda^{[j](i)}\frac{p_{(i)}}{p^{(0)}}, \quad \Lambda^{[0]a} = -\frac{p^a}{m}, \tag{59}$$

$$np \equiv n^\mu p_\mu = -\sqrt{m^2 + \gamma^{ij} p_i p_j}, \quad \gamma^{ik} g_{kj} = \delta^i_j, \tag{60}$$

where $n^\mu = (1, -N^i)/N, \quad n_\mu = (-N, 0, 0, 0)$.

To fix the tetrad field, the so-called time gauge (Schwinger's coining, but introduced by Dirac earlier) proves extremely useful,

$$e^\mu_{(0)} = n^\mu, \quad \text{i.e.} \quad e^0_{(0)} = \frac{1}{N}, \qquad e^i_{(0)} = -\frac{N^i}{N}. \tag{61}$$

Then

$$g_{ij} = e_i^{(m)} e_{(m)j}. \tag{62}$$

The matter Lagrangian density is split into three parts, a kinetic part with time derivative of the matter and field variables, respectively denoted $\mathscr{L}_{MK}$ and $\mathscr{L}_{GK}$, and a constraint part, $\mathscr{L}_{MC}$,

$$\mathscr{L}_{MK}, \quad \mathscr{L}_{GK}, \quad \mathscr{L}_{MC} = -N\mathscr{H}^{\text{matter}} + N^i \mathscr{H}_i^{\text{matter}}. \tag{63}$$

The matter energy and momentum densities read,

$$\mathscr{H}^{\text{matter}} = -np\delta - K^{ij}\frac{p_i nS_j}{np}\delta - (nS^k\delta)_{;k}, \tag{64}$$

$$\mathscr{H}_i^{\text{matter}} = (p_i + K_{ij}nS^j)\delta + \left(\frac{1}{2}\gamma^{mk}S_{ik}\delta + \delta_i^{(k}\gamma^{l)m}\frac{p_k nS_l}{np}\delta\right)_{;m}. \tag{65}$$

The semicolon denotes covariant three-dimensional derivative; later on, the comma will denote partial derivative. The transformation to canonical matter variables, indicated by hats, is given by

$$z^i = \hat{z}^i - \frac{nS^i}{m-np}, \quad nS_i = -\frac{p_k\gamma^{kj}\hat{S}_{ji}}{m}, \tag{66}$$

$$S_{ij} = \hat{S}_{ij} - \frac{p_i nS_j}{m-np} + \frac{p_j nS_i}{m-np}, \tag{67}$$

$$\Lambda^{[i](j)} = \hat{\Lambda}^{[i](k)}\left(\delta_{kj} + \frac{p_{(k)}p^{(j)}}{m(m-np)}\right), \tag{68}$$

$$p_i = \hat{p}_i - K_{ij}nS^j - \hat{A}^{kl}e_{(j)k}e_{l,i}^{(j)} + \left(\frac{1}{2}S_{kj} + \frac{p_{(k}nS_{j)}}{np}\right)\Gamma^{kj}_{\ \ i}, \tag{69}$$

where

$$g_{ik}g_{jl}\hat{A}^{kl} = \frac{1}{2}\hat{S}_{ij} + \frac{mp_{(i}nS_{j)}}{np(m-np)} \tag{70}$$

and

$$S^{ab}S_{ab} = \hat{S}_{(i)(j)}\hat{S}_{(i)(j)} = 2\hat{S}_{(i)}\hat{S}_{(i)} = 2s^2 = \text{const}, \tag{71}$$

$$\hat{\Lambda}^{(i)}_{[k]}\hat{\Lambda}^{[k](j)} = \delta_{ij}, \tag{72}$$

$$d\hat{\theta}^{(i)(j)} \equiv \hat{\Lambda}^{(i)}_{[k]} d\hat{\Lambda}^{[k](j)} = -d\hat{\theta}^{(j)(i)}. \tag{73}$$

Putting $\mathscr{L}_{MK} + \mathscr{L}_{GK} = \hat{\mathscr{L}}_{MK} + \hat{\mathscr{L}}_{GK} + \text{(td)}$, one finds,

$$\hat{\mathscr{L}}_{MK} = \hat{p}_i \dot{\hat{z}}^i \delta + \frac{1}{2} \hat{S}_{(i)(j)} \dot{\hat{\theta}}^{(i)(j)} \delta, \tag{74}$$

$$\hat{\mathscr{L}}_{GK} = \hat{A}^{ij} e_{(k)i} e^{(k)}_{j,0} \delta. \tag{75}$$

Adding the Lagrangian of gravity, $\mathscr{L}_G$, results in a new Lagrangian for gravity, also see Deser and Isham [37],

$$\hat{\mathscr{L}}_{GK} + \mathscr{L}_G = [2\pi^{ij} + \hat{A}^{ij}\delta] e_{(k)i} e^{(k)}_{j,0} \; + \; \mathscr{L}_{GC} - \mathscr{E}_{i,i} \tag{76}$$

with

$$\mathscr{L}_{GC} = -N \mathscr{H}^{\text{field}} + N^i \mathscr{H}^{\text{field}}_i \tag{77}$$

and

$$\mathscr{H}^{\text{field}} = -\frac{1}{\sqrt{g}} \left[g\text{R} + \frac{1}{2} \left(g_{ij} \pi^{ij} \right)^2 - g_{ij} g_{kl} \pi^{ik} \pi^{jl} \right], \tag{78}$$

$$\mathscr{H}^{\text{field}}_i = 2 g_{ij} \pi^{jk}_{;k}, \tag{79}$$

as well as,

$$\mathscr{E}_i = g_{ij,j} - g_{jj,i}. \tag{80}$$

The application of the crucial spatially symmetric time gauge for the tetrads introduced by Kibble, [13],

$$e_{(i)j} = e_{ij} = e_{ji}, \tag{81}$$

$$e_{ij} e_{jk} = g_{ik}, \qquad e_{ij} = \sqrt{(g_{kl})} \quad \text{(matrix root!)}, \tag{82}$$

reduces the tetrads to the metric functions. The partial derivatives of the tetrads result in the expressions

$$e_{(k)i} e^{(k)}_{j,\mu} = B^{kl}_{ij} g_{kl,\mu} + \frac{1}{2} g_{ij,\mu}, \qquad B^{kl}_{ij} = B^{(kl)}_{[ij]}, \tag{83}$$

where $(\cdot\cdot)$ and $[\cdot\cdot]$ denote symmetrization and antisymmetrization, respectively, and

$$2B^{kl}_{ij} = e_{mi}\frac{\partial e_{mj}}{\partial g_{kl}} - e_{mj}\frac{\partial e_{mi}}{\partial g_{kl}}. \tag{84}$$

From the new gravity action the canonical field momentum is easily read off to be

$$\pi^{ij}_{\mathrm{can}} = \pi^{ij} + \frac{1}{2}\hat{A}^{(ij)}\delta + B^{ij}_{kl}\hat{A}^{[kl]}\delta. \tag{85}$$

The ADM spacetime coordinate conditions do take now the following forms,

$$3g_{ij,j} - g_{jj,i} = 0, \qquad \pi^{ii}_{\mathrm{can}} = 0, \tag{86}$$

$$g_{ij} = \Psi^4\delta_{ij} + h^{\mathrm{TT}}_{ij}, \qquad \pi^{ij}_{\mathrm{can}} = \tilde{\pi}^{ij}_{\mathrm{can}} + \pi^{ij\mathrm{TT}}_{\mathrm{can}}, \tag{87}$$

with the transverse-traceless objects h^{TT}_{ij} and $\pi^{ij\mathrm{TT}}_{\mathrm{can}}$,

$$h^{\mathrm{TT}}_{ii} = \pi^{ii\mathrm{TT}}_{\mathrm{can}} = h^{\mathrm{TT}}_{ij,j} = \pi^{ij\mathrm{TT}}_{\mathrm{can},j} = 0, \tag{88}$$

and longitudinal one $\tilde{\pi}^{ij}_{\mathrm{can}}$,

$$\tilde{\pi}^{ij}_{\mathrm{can}} = V^i_{\mathrm{can},j} + V^j_{\mathrm{can},i} - \frac{2}{3}\delta_{ij}V^k_{\mathrm{can},k}. \tag{89}$$

The constraint equations read

$$\mathscr{H}^{\mathrm{field}} + \mathscr{H}^{\mathrm{matter}} = 0, \qquad \mathscr{H}^{\mathrm{field}}_i + \mathscr{H}^{\mathrm{matter}}_i = 0. \tag{90}$$

Finally, the total action in canonical form is given by, [36],

$$W = \int d^4x\, \pi^{ij\mathrm{TT}}_{\mathrm{can}} h^{\mathrm{TT}}_{ij,0} + \int dt \left[\hat{p}_i\dot{\hat{z}}^i + \frac{1}{2}\hat{S}_{(i)(j)}\dot{\hat{\theta}}^{(i)(j)} - E\right], \tag{91}$$

with $E = \oint dS_i\mathscr{E}_i$, and the Hamiltonian reads,

$$E \equiv H_{\mathrm{ADM}} = -8\int d^3x\, \Delta\Psi\left[\hat{z}^i, \hat{p}_i, \hat{S}_{(i)(j)}, h^{\mathrm{TT}}_{ij}, \pi^{ij\mathrm{TT}}_{\mathrm{can}}\right], \tag{92}$$

with Poisson bracket commutation relations

$$\begin{aligned} \{\hat{z}^i, \hat{p}_j\} = \delta_{ij}, \quad \{\hat{S}_{(i)}, \hat{S}_{(j)}\} &= \varepsilon_{ijk}\hat{S}_{(k)}, \\ \{h^{\mathrm{TT}}_{ij}(\mathbf{x},t), \pi^{kl\mathrm{TT}}_{\mathrm{can}}(\mathbf{x}',t)\} &= \delta^{\mathrm{TT}kl}_{ij}\delta(\mathbf{x}-\mathbf{x}'). \end{aligned} \tag{93}$$

5 Post-Newtonian(PN) Hamiltonians

In this section, the spin will be counted of order $(1/c)^0$ and not $1/c$ as in the Introduction. For non-spinning compact objects, the binary dynamics is known up to the 3.5PN order,

$$\begin{aligned} H(t) = &\ m_1c^2 + m_2c^2 + H_N + H_{1PN} \\ &+ H_{2PN} + H_{3PN} + \dots \\ &+ H_{2.5PN}(t) + H_{3.5PN}(t) + \dots, \end{aligned} \tag{94}$$

where the $2.5PN$ and $3.5PN$ Hamiltonians are non-autonomous dissipative ones, [20]. Introducing the following quantities, $\hat{H} = (H - Mc^2)/\mu$, $\mu = m_1m_2/M$, $M = m_1 + m_2$, $\nu = \mu/M$ with $0 \le \nu \le 1/4$ (test particle case $\nu = 0$, equal mass case $\nu = 1/4$), $\mathbf{p} = \mathbf{p}_1/\mu$, $r = r_{12} = |\mathbf{x}_1 - \mathbf{x}_2|$, $p_r = (\mathbf{n}\cdot\mathbf{p})$, $\mathbf{q} = (\mathbf{x}_1 - \mathbf{x}_2)/GM$, and $\mathbf{n} = \mathbf{n}_{12} = \mathbf{q}/|\mathbf{q}|$, in the center-of-mass frame, $\mathbf{p}_1 + \mathbf{p}_2 = 0$, the following expressions hold,

$$\hat{H}_N = \frac{p^2}{2} - \frac{1}{q}, \tag{95}$$

$$c^2\hat{H}_{1PN} = \frac{1}{8}(3\nu - 1)p^4 - \frac{1}{2}[(3+\nu)p^2 + \nu p_r^2]\frac{1}{q} + \frac{1}{2q^2}, \tag{96}$$

$$\begin{aligned} c^4\hat{H}_{2PN} = &\ \frac{1}{16}(1 - 5\nu + 5\nu^2)p^6 \\ &+ \frac{1}{8}[(5 - 20\nu - 3\nu^2)p^4 - 2\nu^2 p_r^2 p^2 - 3\nu^2 p_r^4]\frac{1}{q} \\ &+ \frac{1}{2}[(5+8\nu)p^2 + 3\nu p_r^2]\frac{1}{q^2} - \frac{1}{4}(1+3\nu)\frac{1}{q^3}, \end{aligned} \tag{97}$$

$$\begin{aligned} c^6\hat{H}_{3PN} = &\ \frac{1}{128}(-5 + 35\nu - 70\nu^2 + 35\nu^3)p^8 \\ &+ \frac{1}{16}\Big[(-7 + 42\nu - 53\nu^2 - 5\nu^3)p^6 + (2 - 3\nu)\nu^2 p_r^2 p^4 \\ &\qquad + 3(1-\nu)\nu^2 p_r^4 p^2 - 5\nu^3 p_r^6\Big]\frac{1}{q} \\ &+ \Big[\frac{1}{16}(-27 + 136\nu + 109\nu^2)p^4 + \frac{1}{16}(17 + 30\nu)\nu p_r^2 p^2 \\ &\qquad + \frac{1}{12}(5 + 43\nu)\nu p_r^4\Big]\frac{1}{q^2} \end{aligned}$$

$$+\left[\left(-\frac{25}{8}+\left(\frac{1}{64}\pi^2-\frac{335}{48}\right)\nu-\frac{23}{8}\nu^2\right)p^2\right.$$
$$\left.+\left(-\frac{85}{16}-\frac{3}{64}\pi^2-\frac{7}{4}\nu\right)\nu p_r^2\right]\frac{1}{q^3}$$
$$+\left[\frac{1}{8}+\left(\frac{109}{12}-\frac{21}{32}\pi^2\right)\nu\right]\frac{1}{q^4}. \tag{98}$$

To save space, from the disspative Hamiltonians only the leading one is given, [16],

$$c^5\hat{H}_{2.5PN}(\hat{t})=\frac{2}{5}\left[p_ip_j-\frac{n^in^j}{q}\right]\frac{d^3\hat{Q}_{ij}(\hat{t})}{d\hat{t}^3}. \tag{99}$$

Here $\hat{Q}_{ij}(\hat{t})=\nu(q'^iq'^j-\delta_{ij}q'^2/3)$ and its time derivatives ($\hat{t}=t/GM$) are allowed to be eliminated using the equations of motion. Only after the performance of the phase-space derivatives, the primed variables are allowed to be identified with the unprimed ones.

For convenience, the spin-gravity interaction Hamiltonians are given in the non-center-of-mass frame. To simplify notation, $\mathbf{S}\equiv\hat{\mathbf{S}}$ will be put. The leading order spin-orbit Hamiltonian reads

$$H_{\mathrm{SO}}^{1PN}=\frac{G}{c^2}\sum_a\sum_{b\neq a}\frac{1}{r_{ab}^2}(\mathbf{S}_a\times\mathbf{n}_{ab})\cdot\left[\frac{3m_b}{2m_a}\mathbf{p}_a-2\mathbf{p}_b\right]. \tag{100}$$

The leading order spin(1)-spin(2) Hamiltonian takes the form,

$$H_{\mathrm{S_1S_2}}^{1PN}=\frac{G}{c^2}\sum_a\sum_{b\neq a}\frac{1}{2r_{ab}^3}\left[3(\mathbf{S}_a\cdot\mathbf{n}_{ab})(\mathbf{S}_b\cdot\mathbf{n}_{ab})-(\mathbf{S}_a\cdot\mathbf{S}_b)\right] \tag{101}$$

and the leading order spin(1)-spin(1) dynamics is given by, going beyond linear order in spin,

$$H_{\mathrm{S_1S_1}}^{1PN}=\frac{G}{c^2}\frac{1}{2r_{12}^3}\left[3(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{n}_{12})-(\mathbf{S}_1\cdot\mathbf{S}_1)\right]. \tag{102}$$

The next-to-leading order spin-orbit Hamiltonain reads,

$$H_{SO}^{2PN}=\frac{G}{c^4r^2}\left[-((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})\left[\frac{5m_2\mathbf{p}_1^2}{8m_1^3}+\frac{3(\mathbf{p}_1\cdot\mathbf{p}_2)}{4m_1^2}\right.\right.$$
$$\left.-\frac{3\mathbf{p}_2^2}{4m_1m_2}+\frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})}{4m_1^2}+\frac{3(\mathbf{p}_2\cdot\mathbf{n}_{12})^2}{2m_1m_2}\right]$$
$$+((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12})\left[\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+\frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})}{m_1m_2}\right] \tag{103}$$

$$+ ((\mathbf{p}_1 \times \mathbf{S}_1)\cdot\mathbf{p}_2)\left[\frac{2(\mathbf{p}_2\cdot\mathbf{n}_{12})}{m_1 m_2} - \frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})}{4m_1^2}\right]\Bigg]$$
$$+ \frac{G^2}{c^4 r^3}\Bigg[-((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})\left[\frac{11m_2}{2} + \frac{5m_2^2}{m_1}\right]$$
$$+ ((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12})\left[6m_1 + \frac{15m_2}{2}\right]\Bigg] + (1\leftrightarrow 2) \qquad (104)$$

and the next-to-leading order spin(1)-spin(2) Hamiltonian is given by

$$\begin{aligned}
H^{2PN}_{S_1S_2} = {} & (G/2m_1m_2c^4r^3)[3((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_2\times\mathbf{S}_2)\cdot\mathbf{n}_{12})/2 \\
& + 6((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) \\
& - 15(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
& - 3(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{p}_2) + 3(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) \\
& + 3(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) + 3(\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
& + 3(\mathbf{S}_2\cdot\mathbf{p}_2)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) - 3(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
& + (\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{p}_2) - (\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{p}_1)/2 + (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)/2] \\
& + (3/2m_1^2r^3)[-((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) \\
& + (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{p}_1\cdot\mathbf{n}_{12})] \\
& + (3/2m_2^2r^3)[-((\mathbf{p}_2\times\mathbf{S}_2)\cdot\mathbf{n}_{12})((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12}) \\
& + (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})] \\
& + (6G^2(m_1+m_2)/c^4r^4)[(\mathbf{S}_1\cdot\mathbf{S}_2) - 2(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})]. \qquad (105)
\end{aligned}$$

Finally, the next-to-leading order spin(1)-spin(1) dynamics reads,

$$\begin{aligned}
H^{2PN}_{S_1S_1} = {} & \frac{G}{c^4r^3}\Bigg[-\frac{5m_2}{4m_1^3}(\mathbf{p}_1\cdot\mathbf{S}_1)^2 + \frac{m_2}{m_1^3}\mathbf{p}_1^2\mathbf{S}_1^2 - \frac{21m_2}{8m_1^3}(\mathbf{p}_1\cdot\mathbf{n})^2\mathbf{S}_1^2 \\
& -\frac{3m_2}{8m_1^3}\mathbf{p}_1^2(\mathbf{S}_1\cdot\mathbf{n})^2 + \frac{15m_2}{4m_1^3}(\mathbf{p}_1\cdot\mathbf{n})(\mathbf{S}_1\cdot\mathbf{n})(\mathbf{p}_1\cdot\mathbf{S}_1) - \frac{3}{4m_1m_2}\mathbf{p}_2^2\mathbf{S}_1^2 \\
& +\frac{9}{4m_1m_2}\mathbf{p}_2^2(\mathbf{S}_1\cdot\mathbf{n})^2 - \frac{1}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{S}_1^2 - \frac{9}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{S}_1\cdot\mathbf{n})^2 \\
& +\frac{3}{2m_1^2}(\mathbf{p}_1\cdot\mathbf{S}_1)(\mathbf{p}_2\cdot\mathbf{S}_1) - \frac{3}{2m_1^2}(\mathbf{p}_1\cdot\mathbf{n})(\mathbf{p}_2\cdot\mathbf{S}_1)(\mathbf{S}_1\cdot\mathbf{n}) \\
& -\frac{3}{2m_1^2}(\mathbf{p}_2\cdot\mathbf{n})(\mathbf{p}_1\cdot\mathbf{S}_1)(\mathbf{S}_1\cdot\mathbf{n}) + \frac{15}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{n})(\mathbf{p}_2\cdot\mathbf{n})\mathbf{S}_1^2 \\
& -\frac{15}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{n})(\mathbf{p}_2\cdot\mathbf{n})(\mathbf{S}_1\cdot\mathbf{n})^2\Bigg] \\
& -\frac{G^2m_2}{2c^4r^4}\left[5\left(1+\frac{m_2}{m_1}\right)((\mathbf{S}_1\cdot\mathbf{n})^2 - \mathbf{S}_1^2) + 4\left(1+\frac{2m_2}{m_1}\right)(\mathbf{S}_1\cdot\mathbf{n})^2\right]. \qquad (106)
\end{aligned}$$

Also this Hamiltonian goes beyond linear order in spin. For its derivation an extension of the Tulczyjew stress-energy tensor for pole-dipole particles was needed, [31]. As further examples, the next-to-leading order spin-orbit center-of-energy vector is given,

$$\mathbf{G}^{2PN}_{\rm SO} = -\sum_a \frac{\mathbf{P}_a^2}{8m_a^3}(\mathbf{P}_a\times\mathbf{S}_a)$$
$$+\sum_a\sum_{b\neq a}\frac{m_b}{4m_a r_{ab}}\left[((\mathbf{P}_a\times\mathbf{S}_a)\cdot\mathbf{n}_{ab})\frac{5\mathbf{x}_a+\mathbf{x}_b}{r_{ab}} - 5(\mathbf{P}_a\times\mathbf{S}_a)\right]$$
$$+\sum_a\sum_{b\neq a}\frac{1}{r_{ab}}\left[\frac{3}{2}(\mathbf{P}_b\times\mathbf{S}_a) - \frac{1}{2}(\mathbf{n}_{ab}\times\mathbf{S}_a)(\mathbf{P}_b\cdot\mathbf{n}_{ab})\right.$$
$$\left. -((\mathbf{P}_a\times\mathbf{S}_a)\cdot\mathbf{n}_{ab})\frac{\mathbf{x}_a+\mathbf{x}_b}{r_{ab}}\right], \tag{107}$$

as well as the spin(1)-spin(2) one,

$$\mathbf{G}^{2PN}_{\rm SS} = \frac{1}{2}\sum_a\sum_{b\neq a}\left\{[3(\mathbf{S}_a\cdot\mathbf{n}_{ab})(\mathbf{S}_b\cdot\mathbf{n}_{ab}) - (\mathbf{S}_a\cdot\mathbf{S}_b)]\frac{\mathbf{x}_a}{r_{ab}^3} + (\mathbf{S}_b\cdot\mathbf{n}_{ab})\frac{\mathbf{S}_a}{r_{ab}^2}\right\}. \tag{108}$$

To summarize, for binary systems, and in part for many-body systems too, the following PN Hamiltonians are known fully explicitly,

$$\begin{aligned} H = & \, H_N + H_{1PN} + H_{2PN} + H_{2.5PN} + H_{3PN} + H_{3.5PN} \\ & + H^{1PN}_{SO} + H^{2PN}_{SO} + H^{3PN}_{SO} + H^{3.5PN}_{SO} \\ & + H^{1PN}_{S_1S_2} + H^{2PN}_{S_1S_2} + H^{3PN}_{S_1S_2} + H^{3.5PN}_{S_1S_2} \\ & + H^{1PN}_{S_1S_1} + H^{1PN}_{S_2S_2} + H^{2PN}_{S_1S_1} + H^{2PN}_{S_2S_2} \\ & + H_{p_1S_2^3} + H_{p_2S_1^3} + H_{p_1S_1S_2^2} + H_{p_2S_2S_1^2} \\ & + H_{S_1S_2^3} + H_{S_1^3S_2} + H_{S_1^2S_2^2}. \end{aligned} \tag{109}$$

Also the n-body Hamiltonian through linear order in Newton's gravitational constant G is known $H = H_{1PM}$, as well as the test-spin Hamiltonian in the Kerr metric.

For selfgravitating objects, the Hamiltonians primarily come out in the form $H = H[p,q,h^{\rm TT},\pi_{\rm TT}]$. The transition to a Routhian description of the type $H = H[p,q,h^{\rm TT},\dot{h}^{\rm TT}]$ then allows the derivation of an autonomous Hamiltonian for the conservative dynamics in the form $H = H[p,q,h^{\rm TT}(x;p,q),\dot{h}^{\rm TT}(x;p,q)] = H(p,q)$ as well as a non-autonomous one given by $H(t) = H[p,q,h^{\rm TT}(x;p',q'),\dot{h}^{\rm TT}(x;p',q')] = H(p,q;p',q')$. The presented dynamical systems have found derivations with other methods too. Particularly the Effective Field Theory method by Goldberger and Rothstein, [38], has proven very powerful. For details the reader is referred to the literature.

Acknowledgment

The author thanks Stanley Deser for useful discussions.

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