Increase of Black Hole Entropy in Lanczos-Lovelock Gravity

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Abstract The striking similarity of the laws of black hole mechanics with thermodynamics was first established in case of general relativity (GR). A natural question is to ask whether this analogy is a peculiar property of GR or a robust feature of any generally covariant theory of gravity. We study this question in the context of Lanczos-Lovelock gravity and provide a proof of classical quasi stationary second law.

1 Introduction

General relativity (GR), being quantum mechanically non-renormalizable, may make sense as a Wilsonian effective theory working perturbatively in powers of the dimensionless small parameter $G\,(Energy)^{D-2}$, where G is the D-dimensional Newton's constant. Then the Einstein-Hilbert Lagrangian is the lowest order term (other than the cosmological constant) in a derivative expansion of generally covariant actions for a metric theory, and the presence of higher curvature terms is presumably inevitable. In general, the specific form of these terms will depend on the detailed features of the quantum gravity model. Still, from a purely classical point of view, a natural modification of the Einstein-Hilbert action is to include terms preserving the diffeomorphism invariance and still leading to an equation of motion containing no more than second order time derivatives. Interestingly, this generalization is unique, [1, 2] and goes by the name of Lanczos-Lovelock gravity. Lanczos-Lovelock gravity is free from perturbative ghost [3] and leads to a well-defined initial value formalism [4]. The lowest order Lanczos-Lovelock correction term in space time dimensions $D > 4$, namely the Gauss-Bonnet term, also appears as a low energy α' correction in case of heterotic string theory [3, 5]. Hence, it is interesting to pursue various classical and semi classical properties of Lanczos-Lovelock gravity. For example,

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the striking similarity of the laws of black hole mechanics with thermodynamics was first established in case of general relativity [6] and a natural question is to ask whether this analogy is a peculiar property of GR or a robust feature of any generally covariant theory of gravity. Studying the properties of black holes in a general Lanczos-Lovelock theory may provide a partial answer to this important question.

The equilibrium state version of first law for black holes was established by Wald and collaborators [7, 8] for any arbitrary diffeomorphism invariant theory of gravity. The entropy of the black hole can be expressed as a local geometric quantity integrated over a space-like cross section of the horizon and is associated with the Noether charge of Killing isometry that generates the horizon.

Implicit in the investigations which uses the Wald entropy in these theories is the assumption that the entropy associated with a horizon behaves like ordinary thermodynamic entropy. But, the equilibrium state version of first law for black holes, established by Wald and collaborators [7, 8] requires the existence of a stationary black hole with regular bifurcation surface. As a result, from the equilibrium state version of first law, it is not immediately clear whether the Wald entropy always increases under physical processes, except for black holes in GR, in which the "area theorem" asserts that area of a black hole can not decrease in any process provided null energy condition holds for the matter fields [9]. The area theorem, in turn, follows from Raychaudhuri equation and crucially depends on the contracted Einstein's equation $R_{ab}k^ak^b = 8\pi T_{ab}k^ak^b$ where k^a is the tangent to the horizon. Since the entropy of black holes is no longer proportional to area in Lanczos-Lovelock models of gravity, there is no obvious assurance that the entropy still obeys an increase theorem. As a result, the question of validity of the second law of black hole thermodynamics for arbitrary theory of gravity remains an unresolved issue. Except for the case of $f(R)$-gravity [10], there is no proof of the analog of Hawking's area theorem beyond GR. In the quasi-stationary case, an argument for second law valid for all diffeomorphism invariant gravity theories was given in [10]; but it is based on the assumption that the stationary comparison version of the first law implies the physical process version for quasi-stationary processes.

For the thermodynamic interpretation to be valid, we would expect horizon entropy to increase when a black hole in the Lanczos-Lovelock model participates in some physical process, like, e.g., accretion of matter. Recently, a direct proof of the physical process version of first law is proposed for Einstein-Gauss-Bonnet (EGB) gravity [11] which establishes that the net change of black hole entropy during a physical process is positive as long as matter satisfies null energy condition.
Here, we investigate this question for general Lanczos-Lovelock models and show that during a physical process, the Wald entropy of stationary black holes in general Lanczos-Lovelock gravity monotonically increases provided the matter stress energy tensor obeys null energy condition. As a result, not only the net change of the entropy is positive, but the entropy is increasing at every cross section of the horizon. In this paper, we will present the essential idea and main steps of the cal-

culations. For more details of the derivation, see [12].

Let us start with a brief review of the properties of stationary, non-extremal, Killing horizons. (We adopt the metric signature $(-,+,+,+,...)$ and our sign conventions are the same as those of [13].) In a D-dimensional spacetime, the event horizon is a null hyper-surface $\mathscr{H}$ parametrized by an affine parameter λ. The vector field $k^a = (\partial_\lambda)^a$ is tangent to the horizon and obeys geodesic equation. All $\lambda =$ constant slices are space-like and foliate the horizon. Any point p on such slices has coordinates $\{\lambda, x^A\}$ where x^A, $(A = 2, \cdots, D)$ are the coordinates of a point on $\lambda = 0$ slice connected with p by a horizon generator. We can construct a basis with the vector fields, $\{k^a, l^a, e^a_A\}$ where l^a is a second null vector such that $l^a k_a = -1$. The induced metric on any slice is $\gamma_{ab} = g_{ab} + 2k_{(a}l_{b)}$ and $k^a \gamma_{ab} = 0 = l^a \gamma_{ab}$. The change of the induced metric from one slice to another can be obtained from the metric evolution equation [13],

$$\mathscr{L}_k \gamma_{ab} = 2\left(\sigma_{ab} + \frac{\theta}{(D-2)}\gamma_{ab}\right), \tag{1}$$

where σ_{ab} is the shear and θ is the expansion of the horizon. If the event horizon is also a Killing horizon [1], i.e. the horizon generators are the orbits of a Killing field $\xi^a = (\partial/\partial v)^a$, which is null on the horizon, then the surface gravity κ of the horizon is defined as $\xi^a \nabla_a \xi^b = \kappa \xi^b$. For stationary spacetimes with a Killing horizon, both the expansion and shear vanish and using Raychaudhuri equation and the evolution equation for shear, we obtain [14, 13] that on the horizon,

$$\xi^a \xi^c \gamma^b_i \gamma^d_k R_{abcd} = 0 = R_{ab} \xi^a \xi^b = 0 \tag{2}$$

and

$$\xi^a \gamma^b_i \gamma^c_j \gamma^d_k R_{abcd} = 0. \tag{3}$$

We would like to emphasize that in order to derive these relationships, we have only used the fact that the horizon is a Killing horizon with zero expansion and shear without assuming any further symmetry.

We would like to consider the situation when a stationary black hole is perturbed by a weak matter stress energy tensor and ultimately settles down to a stationary state in the asymptotic future. Since the black hole is stationary in the asymptotic future, the vector field ξ^a is an exact Killing vector at late times. The accretion process is assumed to be slow such that all changes of the dynamical fields are first order in some suitable bookkeeping parameter ε and that we can neglect all viscous effects. More specifically, we assume that, $\theta \sim \sigma_{ab} \sim \mathscr{O}(\varepsilon)$.

In GR, a concrete example of such a physical process is a black hole of mass M slowly accreting matter for a finite time and ultimately settling down to a stationary

[1] Here we make an assumption, that the event horizon of a stationary black hole is also a Killing horizon with regular bifurcation surface. Although this is certainly true for GR, we are not aware of any proof for Lanczos-Lovelock gravity.

state. Then a linearized version of the Raychaudhuri equation gives,

$$\frac{d\theta}{d\lambda} \approx -R_{ab}k^a k^b = -8\,\pi\,T_{ab}k^a k^b, \tag{4}$$

where, we have used Einstein's equation to get the second equality. If the matter stress tensor satisfies null energy condition, i. e. $T_{ab}k^a k^b \geq 0$, the rate of change of the expansion is negative on any slice prior to the asymptotic future. Since the expansion vanishes in the future, the generators must have positive expansion during the accretion process. As a result, the area is monotonically increasing in the physical process. Note that, the result is crucially dependent on the field equation. As a result, the monotonicity of the horizon area is only valid in case of GR. Our aim is to prove a same statement for the Wald entropy during a dynamical change of the black holes in Lanczos-Lovelock gravity.

We shall now turn our attention to the features of Lanczos-Lovelock gravity. As discussed before, a natural generalization of the Einstein-Hilbert Lagrangian is provided by the Lanczos-Lovelock Lagrangian, which is the sum of dimensionally extended Euler densities,

$$\mathscr{L}^D = \sum_{m=0}^{[D-1)/2]} \alpha_m \mathscr{L}_m^D, \tag{5}$$

where the α_m are arbitrary constants and $\mathscr{L}_m^D$ is the m-th order Lanczos-Lovelock term given by,

$$\mathscr{L}_m^D = \frac{1}{16\pi} \sum_{m=0}^{[D-1)/2]} \frac{1}{2^m} \delta^{a_1 b_1 \ldots a_m b_m}_{c_1 d_1 \ldots c_m d_m} R^{c_1 d_1}_{a_1 b_1} \cdots R^{c_m d_m}_{a_m b_m}, \tag{6}$$

where R^{cd}_{ab} is the D dimensional curvature tensor and the generalized alternating tensor $\delta^{\cdots}_{\cdots}$ is totally anti-symmetric in both set of indices. The Einstein-Hilbert Lagrangian is a special case of Eq. (6) when $m = 1$. The field equation of Lanczos-Lovelock theory is, $G_{ab}/(16\pi) + \alpha_m E_{(m)ab} = (1/2)T_{ab}$ where,

$$E^i_{(m)j} = -\frac{1}{16\pi} \frac{1}{2^{m+1}} \delta^{i a_1 b_1 \ldots a_m b_m}_{j c_1 d_1 \ldots c_m d_m} R^{c_1 d_1}_{a_1 b_1} \cdots R^{c_m d_m}_{a_m b_m}, \tag{7}$$

and $m \geq 2$. For convenience, we have written the GR part (i.e. for $m = 1$) separately so that the GR limit can be easily verified by setting all α_m's to zero. Spherically symmetric black hole solutions in Lanczos-Lovelock gravity was derived in [15, 16] and the Wald entropy associated with a stationary Killing horizon is [17, 18, 19],

$$S = \frac{1}{4} \int \rho\, \sqrt{\gamma}\, dA, \tag{8}$$

where the entropy density

$$\rho = \left(1 + \sum_{m=2}^{[D-1)/2]} 16\pi m \alpha_m \,{}^{(D-2)}L_{(m-1)}\right). \tag{9}$$

The integration is over $(D-2)$-dimensional space-like cross-section of the horizon and ${}^{(D-2)}L_{(m-1)}$ is the intrinsic $(m-1)$-th Lanczos-Lovelock scalar of the horizon cross-section. We would like to prove that this entropy always increases when a black hole is perturbed by a weak matter stress energy tensor of $\mathscr{O}(\varepsilon)$ provided the matter obeys null energy condition.

The change in entropy is [10],

$$\Delta S = \frac{1}{4}\int_{\mathscr{H}} \left(\frac{d\rho}{d\lambda} + \theta\,\rho\right) d\lambda\,\sqrt{\gamma}dA. \tag{10}$$

We define a quantity Θ as,

$$\Theta = \left(\frac{d\rho}{d\lambda} + \theta\,\rho\right). \tag{11}$$

In case of GR, Θ is equal to the expansion parameter of the null generators. But, in case of a general gravity theory, Θ is the rate of change of the entropy associated with a infinitesimal portion of horizon (see [10] for similar construction in $f(R)$ gravity). We would like to prove that given null energy condition holds, Θ is positive on any slice in a physical process.

In order to proceed, we would like to study the rate of change of Θ along the congruence using Raychaudhuri equation and the evolution equation of shear [13]. We are only interested in quantities first order in perturbation over a background stationary spacetime. Therefore, when we encounter a product of two quantities X and Y, to extract the part linear in perturbation, we will always express such a product as,

$$XY \approx X^{(B)}Y^{(P)} + X^{(P)}Y^{(B)}, \tag{12}$$

where $X^{(B)}$ is the value of the quantity X evaluated on the stationary background and $X^{(P)}$ is the perturbed value of X linear in perturbation. Note that, on the stationary background, Raychaudhuri equation demands $R^{(B)}_{ab}k^ak^b = 0$ and since $T^{(B)}_{ab}k^ak^b = 0$, we have $E^{(B)}_{(m)ab}k^ak^b = 0$. Also, to simplify the calculation, we use diffeomorphism freedom to make the null geodesic generators of the event horizon of the perturbed black hole coincide with the null geodesic generators of the background stationary black hole [20].

Using the perturbation scheme mentioned above and the evolution equation of θ and σ_{ab} to linear order as $d\theta/d\lambda \approx -R^{(P)}_{ab}k^ak^b$ and $d\sigma_{ab}/d\lambda \approx C^{(P)}_{acdb}k^ck^d$ and further using conditions Eq.(2) and Eq.(3) on the stationary background, the evolution equation of Θ to linear order in perturbation can be written as [12],

$$\frac{d\Theta}{d\lambda} = -8\pi T_{ab}k^ak^b + \mathscr{O}(\varepsilon^2), \tag{13}$$

Eq.(13) shows that if the null energy condition holds, the rate of change of Θ is always negative during a slow classical dynamical process (i.e. ignoring the terms which are higher order in the perturbation) which perturbs the black hole and leads to a new stationary state. Since, the final state is assumed to be stationary, both θ and σ and as a consequence, Θ vanishes in the asymptotic future. Hence, we can use the same argument as with the expansion parameter in case of GR to conclude that Θ must be positive at every slice during the physical process. As a result, we conclude that the horizon entropy of black holes in Lanczos-Lovelock gravity is a monotonically increasing function during any quasi-stationary physical process, i.e.

$$\frac{dS}{d\lambda} \geq 0. \tag{14}$$

which is what we set out to prove.
In case of a dynamical scenario, it is possible to write down several candidates for the black hole entropy beyond GR [21], such that all the expressions have the same stationary limit. We have actually chosen a particular expression and the validity of Eq.(14) favors such a choice. In fact, in ref. [8], a local and geometrical prescription for the entropy of dynamical black holes is proposed. This proposal is based on a boost invariant construction and agrees with the Wald's Noether charge formula for stationary black holes and their perturbations. Interestingly, for Lanczos-Lovelock gravity, the entropy expression used in this work matches the expression obtained from the boost invariant construction. Consequently, our result provides a strong justification in favor of the prescription for dynamical entropy as proposed in ref. [8]. This may also be important to decide the right candidate for the entropy of non stationary black holes for non Lanczos-Lovelock gravity models. Also, one would like to relax the quasi-stationarity physical process assumption and calculate the full change of the Wald entropy along the horizon to understand the validity of classical second law for the Lanczos-Lovelock gravity.

The last point is related to the special status enjoyed by Lanczos-Lovelock models. The derivation presented here, used identities which are very specific to Lanczos-Lovelock models and do not generalize to an arbitrary theory of gravity. Therefore, it would be worthwhile to find a general approach which can answer whether classical second law holds in a physical process for any diffeomorphism invariant gravity theory or applies to a special class of action functionals. This may be useful as a criterion to select a sub class of diffeomorphism invariant actions as preferred theories where a consistent formulation of black hole thermodynamics is possible.

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