Instability of anti-de Sitter spacetime

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Abstract In this talk we summarize our recent numerical and perturbative calculations which indicate that AdS spacetime is unstable. Namely, we study spherically symmetric Einstein-massless-scalar field equations with negative cosmological constant and show that this system is unstable against black hole formation for a large class of initial data arbitrarily close to the AdS solution. We conjecture that this instability is triggered by a resonant mode mixing which gives rise to diffusion of energy from low to high frequencies.

Introduction

Anti-de Sitter (AdS) spacetime is the unique maximally symmetric solution of the vacuum Einstein equations $G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0$ with negative cosmological constant Λ. In $d+1$ dimensions the AdS metric in global dimensionless coordinates ($t \in \mathbb{R}, x \in [0,\pi/2), \omega \in S^{d-1}$) reads

$$ds^2 = \frac{\ell^2}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x\, d\omega^2_{S^{d-1}}\right),$$

where $\ell^2 = -d(d-1)/2\Lambda$ sets the length scale. Conformal infinity $x = \pi/2$ is the timelike cylinder $\mathscr{I} = \mathbb{R} \times S^{d-1}$ with the boundary metric $ds^2_{\mathscr{I}} = -dt^2 + \sin^2 x\, d\omega^2_{S^{d-1}}$.

Asymptotically AdS spacetimes (that is spacetimes which share the conformal boundary with AdS but may be very different in the bulk, in particular may con-

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tain horizons) have come to play a central role in theoretical physics, prominently due to the AdS/CFT correspondence which conjectures a duality between gravity in the AdS bulk and a quantum field theory on the conformal boundary at infinity. By the positive energy theorem, AdS spacetime is a ground state among asymptotically AdS spacetimes, much as Minkowski spacetime is a ground state among asymptotically flat spacetimes. However, the evolutions of small perturbations of these ground states are different. In the case of Minkowski, small perturbations disperse to infinity and the spacetime is asymptotically stable [1]. In contrast, asymptotic stability of AdS is precluded because the conformal boundary acts like a mirror at which perturbations propagating outwards bounce off and return to the bulk that results in complex nonlinear wave interactions in an effectively bounded domain. Understanding of these interactions is the key to the problem of stability of AdS spacetime.

Instability of anti-de Sitter spacetime

In this talk we summarize our recent numerical and perturbative calculations [2, 3] which indicate that AdS spacetime is unstable. To make the problem tractable we assume spherical symmetry. Since by Birkhoff's theorem spherically symmetric vacuum solutions are static, we need to add matter to generate dynamics. A simple matter model is the minimally coupled massless scalar field

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G \left(\partial_\alpha \phi\, \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} (\partial\phi)^2\right), \qquad g^{\alpha\beta}\nabla_\alpha \nabla_\beta \phi = 0. \qquad (1)$$

Let us recall that in the asymptotically flat case ($\Lambda = 0$) this model has led to important insights, such as the proof of the weak cosmic censorship by Christodoulou [4, 5] and the discovery of critical phenomena at the threshold for black hole formation by Choptuik [6].

We use the following parametrization of asymptotically AdS spacetimes

$$ds^2 = \frac{\ell^2}{\cos^2 x}\left(-Ae^{-2\delta}dt^2 + A^{-1}dx^2 + \sin^2 x\, d\omega^2_{S^{d-1}}\right)$$

where A and δ are functions of (t,x). As our results are qualitatively the same for all $d \geq 3$, for concreteness we set $d = 3$ herefater. Under the above assumptions the Einstein-massless scalar field equations (1) reduce to the quasilinear hyperbolic-elliptic system consisting of the scalar wave equation

$$\partial_t \left(A^{-1} e^{\delta} \partial_t \phi\right) = \frac{1}{\tan^2 x} \partial_x \left(\tan^2 x\, A\, e^{-\delta} \partial_x \phi\right) \qquad (2)$$

and two constraint equations (we set $4\pi G = 1$)

$$\partial_x A = \frac{1+2\sin^2 x}{\sin x \cos x}(1-A) - \sin x \cos x A \rho, \qquad \partial_x \delta = -\sin x \cos x \rho, \tag{3}$$

where

$$\rho = A^{-2} e^{2\delta} (\partial_t \phi)^2 + (\partial_x \phi)^2$$

is the scalar field energy density. This system has a one-parameter family of static solutions ($\phi = 0, \delta = 0, A = 1 - M\cos^3 x/\sin x$) which are Schwarzschild-AdS black holes for $M > 0$ and the pure AdS for $M = 0$.

We restrict our attention to smooth solutions with finite mass

$$M := \frac{1}{2}\int_0^{\pi/2} A\rho \tan^2 x \, dx.$$

It follows that near $x = \pi/2$ we must have (using $y = \pi/2 - x$)

$$\phi(t,x) = f_\infty(t) y^3 + \mathscr{O}\left(y^5\right), \;\; \delta(t,x) = \delta_\infty(t) + \mathscr{O}\left(y^6\right), \;\; A(t,x) = 1 - 2My^3 + \mathscr{O}\left(y^6\right).$$

The local well-posedness of the above initial-boundary value problem was proved in [7].

The dynamics of solutions starting from small initial data

$$(\phi, \dot\phi)_{|t=0} = (\varepsilon f(x), \varepsilon g(x))$$

can be approximated using weakly nonlinear perturbation analysis. To this end we expand the solution in the perturbation series

$$\phi = \varepsilon \phi_1 + \varepsilon^3 \phi_3 + \ldots, \quad \delta = \varepsilon^2 \delta_2 + \varepsilon^4 \delta_4 + \ldots, \quad 1 - A = \varepsilon^2 A_2 + \varepsilon^4 A_4 + \ldots$$

where $(\phi_1, \dot\phi_1)_{|t=0} = (f(x), g(x))$ and $(\phi_j, \dot\phi_j)_{|t=0} = (0,0)$ for $j > 1$. Inserting this expansion into the field equations (2) and (3) and collecting terms of the same order in ε, we obtain a hierarchy of linear equations which can be solved order-by-order. At the first order we get the linear wave equation

$$\ddot\phi_1 + L\phi_1 = 0,$$

where

$$L = -\frac{1}{\tan^2 x}\,\partial_x\left(\tan^2 x \, \partial_x\right)$$

is an essentially self-adjoint operator on $L^2([0,\pi/2], \tan^2 x \, dx)$. The eigenvalues of L are $\omega_j^2 = (3+2j)^2$ ($j = 0, 1, \ldots$) which implies that AdS is linearly stable. The corresponding orthonormal eigenfunctions are

$$e_j(x) = d_j \cos^3 x \, P_j^{\left(\frac{1}{2},\frac{3}{2}\right)}(\cos 2x),$$

where d_j is a normalization factor. Thus, at the linear level the solution is

$$\phi_1(t,x) = \sum_{j=0}^{\infty} a_j \cos(\omega_j t + \beta_j)\, e_j(x),$$

where amplitudes a_j and phases β_j are determined by the initial data. Using this solution at the second order we get perturbations of the metric functions A_2 and δ_2 (so called backreaction) and at the third order we obtain an inhomogeneous linear wave equation of the form $\ddot{\phi}_3 + L\phi_3 = S(\phi_1, A_2, \delta_2)$. A calculation shows that in general ϕ_3 contains secular terms that grow linearly in time. They are due to four-wave resonances present in the Fourier decomposition of the source S. We interpret this breakdown of the perturbation analysis as indicating the onset of instability at time of order $\mathscr{O}(\varepsilon^{-2})$. We believe that the secular terms appearing in ϕ_3 are progenitors of the higher-order resonant mode mixing which shifts the energy spectrum to higher frequencies. This heuristics is corroborated by numerical simulations which show that, indeed, generic perturbations start to grow rapidly after a time that scales as ε^{-2}. This growth eventually leads to the formation of a horizon.

To demonstrate the transfer of energy to higher frequencies we define the Fourier coefficients

$$\Phi_j := (A^{1/2}\, \partial_x \phi, e_j') \quad \text{and} \quad \Pi_j := (A^{-1/2} e^{\delta}\, \partial_t \phi, e_j)$$

and express the mass as the Parseval sum

$$M = \sum_{j=0}^{\infty} E_j(t),$$

where

$$E_j := \Pi_j^2 + \omega_j^{-2} \Phi_j^2$$

is the j-mode energy. The evolution of the energy spectrum, that is the distribution of mass among the modes, is depicted in Fig. 1 for gaussian initial data. Initially, the energy is concentrated in low modes; the exponential cutoff of the spectrum expresses the smoothness of initial data. During the evolution the range of excited modes increases and the spectrum becomes broader. Just before horizon formation the spectrum exhibits the power-law scaling $E_j \sim j^{-\alpha}$ with exponent $\alpha \approx 1.2$. This value seems to be universal, i.e., the same for all initial data (but it changes with dimension d). Note that the formation of a black hole provides a cutoff for the turbulent energy cascade (in amusing analogy to viscosity for the turbulent cascade in fluids). Clearly, the formation of the power-law spectrum reflects the loss of smoothness of the solution during collapse; it would be very interesting compute α analytically.

To summarize, our numerical simulations and formal nonlinear perturbation analysis lead us to conjecture that anti-de Sitter space is unstable against the formation of a black hole under arbitrarily small generic perturbations. We wish to stress the genericity condition in the above conjecture: we do not claim that *all* perturbed solutions end up as black holes. On the contrary, in [2] we gave evidence for the existence of non-generic solutions that remain non-singular for very long (possibly in-

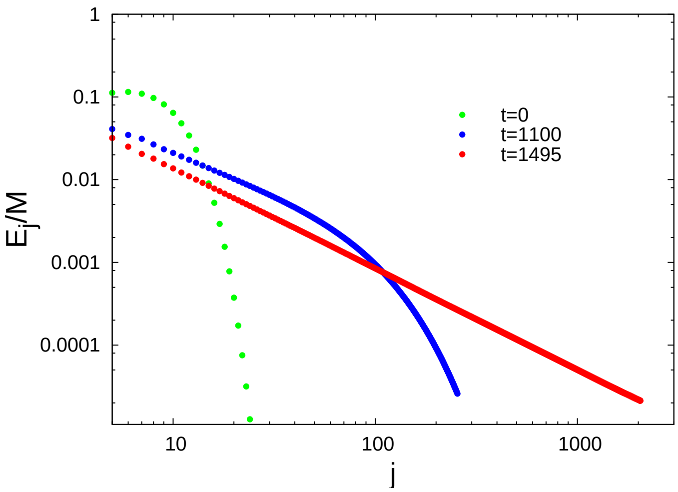

Fig. 1 Log-log plot of the energy spectrum at three moments of time: initial, intermediate, and just before collapse. The fit of the power law $E_j \sim j^{-\alpha}$ at time $t = 1495$ gives the slope $\alpha \approx 1.2$.

finite) time. In particular, preliminary calculations based on the Poincaré-Lindstedt method indicate the existence of time periodic solutions. A similar conjecture (existence of geons) was put forward by Dias, Horowitz and Santos [8] for the vacuum Einstein equations.

The results described above have opened up new and unexpected research paths lying at the interface of classical general relativity and turbulence theory. Exploration of these paths will hopefully lead to better understanding of the dynamics of asymptotically AdS spacetimes, which in turn may have interesting implications in gauge/gravity dualities.

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