Superradiance or Total Reflection?

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Abstract Numerical evolution of massless scalar fields on Kerr background is studied. The initial data is chosen to have compact support separated from the ergoregion and to yield nearly monochromatic incident wave packets. The initial data is also tuned to maximize the effect of superradiance. We give evidence indicating that instead of the anticipated energy extraction from the black hole the incident radiation fails to reach the ergoregion and instead it suffers a nearly perfect reflection.

1 Introduction

To motivate our investigations let us mention first that the stability of the Kerr family of black hole solutions within the space of the vacuum solutions to the Einstein equations is one of the most important unresolved issues in general relativity. The ultimate goal is to provide boundedness and decay statements for solutions of the vacuum Einstein equations around the members of the Kerr family.

It may be a surprise that—even nowadays when numerical simulations of binary black hole systems become a daily routine—essentially all work concerning the aforementioned black hole stability problem has been confined to the linearized setting. Indeed, considerations are restricted to study of the solutions to the Klein-Gordon equation

$$\Box_K \Phi = 0 \tag{1}$$

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on Kerr background. This is done with the hope that the understanding of these simplified scalar perturbations is a good preparation to the study of the more complicated problem of complete, but yet linear, gravitational perturbations.

It should also be mentioned that all the available analytic proofs justifying the linear stability, with respect to scalar perturbations subject to (1), are known to be restricted to the case of slowly rotating subextremal Kerr black holes [1, 2].

2 Superradiance

Recall first that superradiance, as a new phenomenon, was discovered in the early 70's and it may be associated with the names of Misner, Zel'dovich and Starobinskii [3, 4, 5]. It is also considered to be the wave analog of the Penrose process and it is supposed to allow energy to be extracted from black holes.

The common belief related to the interaction of black holes with incident radiation is summarized as "...if scalar, electromagnetic or gravitational wave is incident upon a black hole, part of the wave (the "transmitted wave") will be absorbed by the black hole and part of the wave (the "reflected wave") will escape to infinity" [6]. Recall that by using Teukolsky's equation [7] the evolution of scalar, electromagnetic and gravitational perturbations can be investigated within the same setting. It is also important to be mentioned that all the conventional arguments ending up with superradiance, including the ones based on Teukolsky's equation, refer to properties of individual modes [8].

Interestingly, as first pointed out by Bekenstein [9], whenever superradiance occurs it can be seen to be completely consistent with the laws of black hole thermodynamics. It is also worth to mention some of the expectations concerning scalar perturbations. As claimed in [10]: "Starobinskii made an asymptotic expansion for the reflection coefficient and found a relative gain of energy of about 5% for $m = 1$ and less than 1% for $m \geq 2$."

3 Superradiance in mode analysis

It was realized first by Carter [11] that the temporal Fourier transform, $\mathscr{F}\Phi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \Phi\, e^{i\omega t} dt$, of a solution to (1) may be decomposed as

$$\mathscr{F}\Phi(\omega, r_*, \vartheta, \varphi) = \frac{1}{\sqrt{r^2+a^2}} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} R^m_{\ell,\omega}(r_*) S^m_{\ell,a\omega}(\vartheta, \varphi)\,, \tag{2}$$

where $t, r_*, \vartheta, \varphi$ are local coordinates, while ω is the frequency in the time translation direction. In (2) $S^m_{\ell,a\omega}$ denotes the *oblate spheroidal harmonic functions*, with oblateness parameter $a\omega$, and with angular momentum quantum numbers ℓ, m. The functions $S^m_{\ell,a\omega}$ are eigenfunctions of a self-adjoint operator.

For the radial functions $R^m_{\ell,\omega}$ in (2) a *one-dimensional Schrödinger equation* of the form

$$\frac{d^2R^m_{\ell,\omega}}{dr_*^2}+\left[\left(\omega-\frac{ma}{r^2+a^2}\right)^2+(r-r_H)\cdot V^m_{\ell,\omega}(r_*)\right]R^m_{\ell,\omega}=0\,, \tag{3}$$

can be derived from (1), with suitable real potentials $V^m_{\ell,\omega}(r_*)$.

The "physical solutions" to (3) are supposed to possess the asymptotic behavior

$$R^m_{\ell,\omega}\sim\begin{cases}e^{-i\omega r_*}+\mathscr{R}\,e^{+i\omega r_*} & \text{as } r\to\infty\\ \mathscr{T}\,e^{-i(\omega-m\Omega_H)r_*} & \text{as } r\to r_H\,,\end{cases} \tag{4}$$

where Ω_H stands for the angular velocity of the black hole with respect to the asymptotically stationary observers, while $\mathscr{R}$ and $\mathscr{T}$ denote the reflection and transmission coefficients. Notice that these boundary conditions presume the existence of a transmitted wave submerging into the ergoregion.

By evaluating the Wronskian of the associated fundamental solutions, "close" to infinity and "close" to the horizon, the relation

$$(\omega-m\Omega_H)\,|\mathscr{T}|^2=(1-|\mathscr{R}|^2)\,\omega\,. \tag{5}$$

can be seen to hold. In virtue of this relation it follows then that whenever $|\mathscr{R}|>1$—or equivalently whenever $|\mathscr{T}|$ does not vanish and the inequality

$$0<\omega<m\,\Omega_H \tag{6}$$

is satisfied—energy is supposed to be acquired by the backscattered scalar mode due to its interaction with the Kerr black hole.

4 Numerical studies of superradiance

So far our considerations have been restricted to the study of individual modes. However the investigation of the linear stability problem [12, 13, 1, 2] taught us the lesson that statements which are valid at the level of individual modes typically do not imply statements for finite energy solutions composed of infinitely many modes.

This section is to reveal some of our pertinent numerical results. Before proceeding let us mention that the first time domain studies of superradiance were carried out long time ago in [14, 15]. The scale of energy extraction was found to be smaller than the estimates recalled above. The numerical results reported below were derived by making use of our code called GridRipper which is fully spectral in the angular directions while the dynamics in the complementary 1+1 Lorentzian spacetime is followed by making use of a fourth order finite differencing scheme [16, 17, 18].

4.1 The initial data

To have an incident scalar wave—to study the way a to be superradiant solution acquires extra energy by submerging into the ergoregion—in a sufficiently small neighborhood of the initial data surface in the asymptotic region, the solution was assumed to posses the form

$$\Phi(t,r_*,\vartheta,\tilde{\varphi}) \approx e^{-\mathrm{i}\,\omega_0\,(r_*-r_{*0}+t)} f(r_*-r_{*0}+t)\, Y_\ell^m(\vartheta,\tilde{\varphi}), \tag{7}$$

where $f:\mathbb{R}\to\mathbb{C}$ is a smooth function of compact support and ω_0, r_{*0} are real parameters. This suggest the use of initial data

$$\begin{aligned} \phi(r_*,\vartheta,\tilde{\varphi}) &= e^{-\mathrm{i}\,\omega_0\,(r_*-r_{*0})} f(r_*-r_{*0})\, Y_\ell^m(\vartheta,\tilde{\varphi})\,, \\ \phi_t(r_*,\vartheta,\tilde{\varphi}) &= -\mathrm{i}\,\omega_0\,\phi(r_*,\vartheta,\tilde{\varphi}) + e^{-\mathrm{i}\,\omega_0\,(r_*-r_{*0})} f'(r_*-r_{*0})\, Y_\ell^m(\vartheta,\tilde{\varphi})\,, \end{aligned}$$

where f' denotes the first derivative of $f:\mathbb{R}\to\mathbb{C}$. The Fourier transform, $\mathscr{F}\Phi$, of the approximate solution (7) reads

$$\mathscr{F}\Phi(\omega,r_*,\vartheta,\tilde{\varphi}) \approx e^{-\mathrm{i}\,\omega\,(r_*-r_{*0})} \mathscr{F}f(\omega-\omega_0)\, Y_\ell^m(\vartheta,\tilde{\varphi}), \tag{8}$$

where ω is the temporal frequency and $\mathscr{F}f$ stands for the Fourier-transform of f. Notice that $\mathscr{F}f$ plays the role of a frequency profile, which guaranties that whenever $\mathscr{F}f$ is chosen to be sufficiently narrow the approximate solution (7) has to be close to a monochromatic wave packet, which for suitable value of ω_0 becomes superradiant (for more details see [17]).

4.2 Numerical results

The plots shown below refer to the evolution of pure quadrupole type initial data with a radial profile function $f:\mathbb{R}\to\mathbb{C}$

$$f_w(x) = \begin{cases} e^{\left[-\left|\frac{w}{x+\frac{w}{2}}\right| - \left|\frac{w}{x-\frac{w}{2}}\right| + 4\right]}\,, \ \mathrm{if} x\in[-\frac{w}{2},\frac{w}{2}] \\ \qquad\qquad 0\,, \ \mathrm{otherwise}\,, \end{cases} \tag{9}$$

which is a smooth function of the real variable x with compact support $[-\frac{w}{2},\frac{w}{2}]$, and with initial parameters $M=1$, $a=0.99$, $\ell=m=2$, $\omega_0=\frac{1}{2}m\Omega_H$, $r_{*0}=31.823$.

It is important to be sure (see Fig.1) that the above choice yields a solution with the expected frequency profile.

The time dependence of the radial energy and angular momentum distributions, along with the complete power spectrum, are shown[1] in Fig.2 and 3 below. These

[1] Note that on all the included 2-dimensional plots the indicated quantities are integrated with respect to the radial degrees of freedom.

$\omega/(m\Omega_H)$

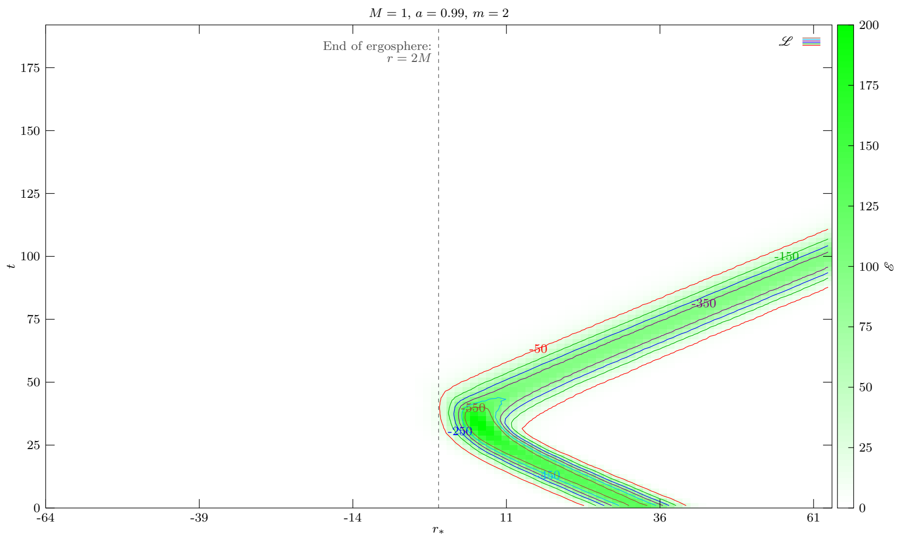

Fig. 1 The power spectrum of the to be superradiant solution at $r_* = 14$, located between the compact support and the black hole.

figures indicate that the reported nearly perfect reflection really does happen for the considered to be superradiant solution.

Fig. 2 The radial energy and angular momentum distributions.

Fig. 3 The radial distribution of the power spectrum.

It is also informative to have a look at the corresponding figures (see Fig.4 and 5) for an almost to be superradiant solution yielded by shifting the compact support towards the black hole—decreasing thereby slightly the ratio of the angular momentum of the radiation to its energy—while all the other parameters were kept intact. Notice that, in virtue of Fig.5, the frequency content of the part of the incident wave packet, submerging into the ergoregion, gets completely evacuated from the superradiant domain.

It is also important to be mentioned that to the accuracy of our code, no energy extraction—or, at least, not more than 10^{-3} times the initial energy—happened in either of these (or analogous) simulations.

5 Summary

The numerical evolution of massless Klein-Gordon field on Kerr background, arising from initial data with compact support in the asymptotic region, was considered. The incident wave packet was tuned to maximize the effect of superradiance.

For perfectly tuned initial data no energy extraction could be observed. Significant part of the incident radiation fails to reach the ergoregion and the time evolution mimics the phenomenon of a total reflection.

To get some insight about the physical mechanism beyond the reported nearly total reflection it turned out to be useful to compare the energy and angular momentum content of the initial data. Fig.6 is to demonstrate that far too much angular

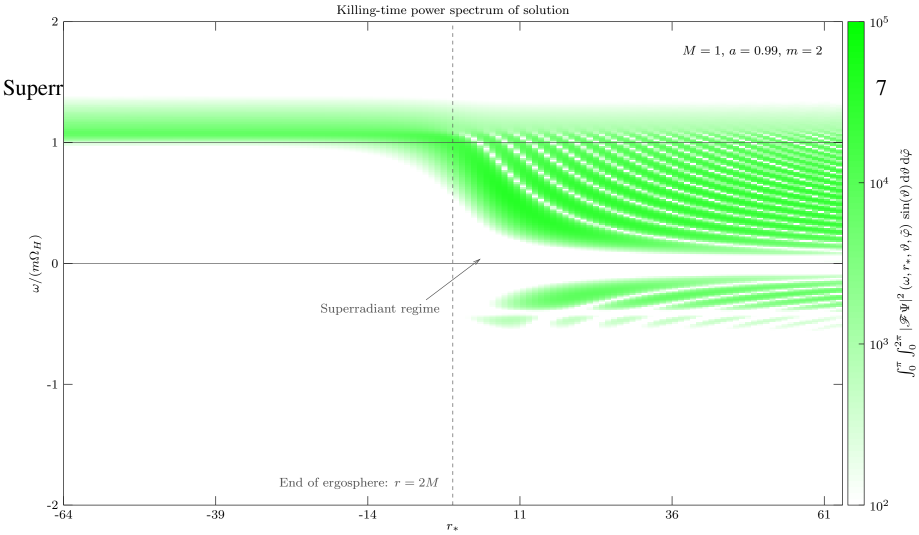

Fig. 4 The frequency ω_0 of this nearly superradiant solution is the same as before. Only the support is shifted to get a submerging part.

Fig. 5 The untuned solution has a submerging part but its power spectrum jumps out of the superradiant domain on reaching the ergoregion.

momentum is stored by the to be superradiant wave packets, as $dE < \Omega_H \, dL$ holds for them, which—in virtue of the second law of black hole thermodynamics—does not allow these packets to enter the black hole region. Accordingly, the observed

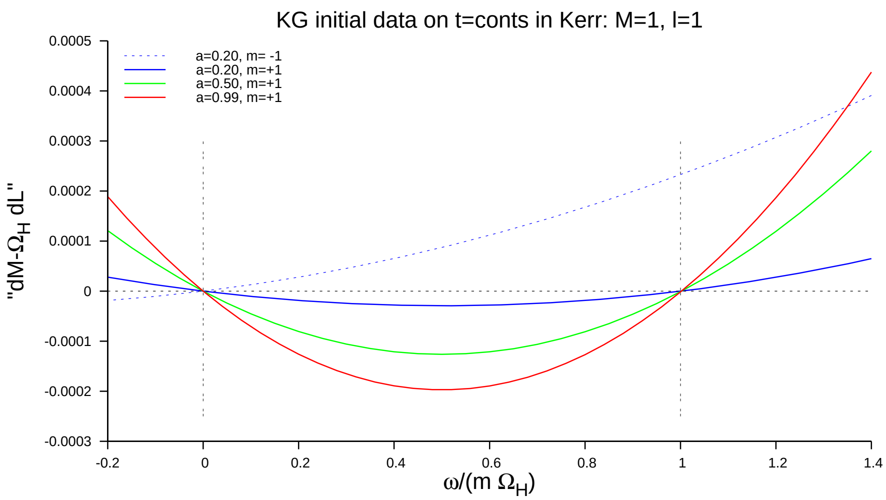

Fig. 6 The energy and angular momentum content of the initial data are compared. The to be superradiant configurations cannot deliver their full energy and angular momentum to the black hole without violating the second law of black hole thermodynamics.

nearly total reflection may be considered as the field theoretical analog of the phenomenon in Wald's thought experiments [19] demonstrating, in the early 70', that a Kerr black hole does not capture a particle that would cause a violation of the relation $m^2 \geq a^2 + e^2$.

Acknowledgments

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