On a Five-Dimensional Version of the Goldberg-Sachs Theorem

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Abstract The recently developed generalization of the Goldberg-Sachs theorem to five-dimensional Einstein spacetimes is summarized. This generalization involves two steps. First it has been proven that in arbitrary dimension an Eistein spacetime admitting a multiple WAND admits also a multiple *geodetic* WAND. Second, in five dimensions, the 3×3 optical matrix of such geodetic multiple WAND can be cast to one of three canonical forms, each determined by two free parameters.

1 Introduction

Recently a generalization of the Petrov classification of the Weyl tensor to arbitrary dimension was developed [1] (see [2] for a recent review). It is thus natural to ask whether the four dimensional Goldberg-Sachs (GS) theorem can be in some form extended to higher dimensions.

In four dimensions GS theorem proved to be a very useful tool for studying and constructing new, algebraically special solutions of the Einstein equations, e.g. the Kerr solution [3]. It states that *in a (non-conformally flat) Einstein spacetime*[1]*, a null vector field is a repeated principal null direction (of the Weyl tensor) if, and only if, it is geodetic and shear-free*.

A higher dimensional analogue of the principal null direction (PND) of the Weyl tensor is a so called Weyl Aligned Null Direction (WAND) which in four dimen-

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[1] An Einstein spacetime is a solution of the vacuum Einstein equation, possibly with a cosmological constant, i.e. with the Ricci tensor $R_{ab} = (R/d)g_{ab}$ in d dimensions.

sions coincides with PND. We will say that a higher dimensional spacetime is algebraically special if it admits a *multiple* WAND (an analogue of a repeated PND).

In contrast with the four-dimensional case, higher dimensional Einstein spacetimes may admit (only in type D spacetimes) *non-geodetic* multiple WANDs. However, it has been shown in [4] that such spacetimes also always admit *geodetic* multiple WANDs. The higher dimensional generalization of the "geodetic" part of the GS theorem thus reads [4]:

Proposition 1. *An Einstein spacetime admits a multiple WAND if, and only if, it admits a geodetic multiple WAND.*

Therefore when considering algebraically special Einstein spacetimes without loss of generality one can always choose a *geodetic* multiple WAND.

For the formulation of the "shearfree" part of the GS theorem we need to introduce the $(d-2)\times(d-2)$ *optical matrix* ρ_{ij}

$$\rho_{ij} \equiv m^{\mu}_{(i)} m^{\nu}_{(j)} \nabla_{\nu} \ell_{\mu}, \tag{1}$$

corresponding to a geodetic null vector field $\boldsymbol{\ell}$ coinciding with a multiple WAND and with $m_{(i)}$ being orthonormal spacelike vectors orthogonal to $\boldsymbol{\ell}$.

In contrast with the four-dimensional case, algebraically special Einstein spacetimes may admit multiple WANDs with non-vanishing shear[2] and in fact in the generic case shear is non-vanishing. Therefore a generalization of *necessary*[3] conditions on ρ_{ij} following from the existence of multiple WAND will not be straightworward and in the next section we will thus limit ourselves to the case of five dimensions.

Let us conclude this section with mentioning some *special* classes of spacetimes where the *necessary* conditions on ρ_{ij} following from the multiple WAND condition have been known in any dimensions.

1.1 Necessary conditions on ρ_{ij} for various special classes of spacetimes

1.1.1 Types N and III

The multiple WAND in vacuum spacetimes of type N must be geodetic, the optical matrix must have rank 2 and it can be put into a form [7]

$$\boldsymbol{\rho} = b \,\mathrm{diag}\left(\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}, 0, \ldots, 0 \right). \tag{2}$$

[2] Shear is defined as traceless symmetric part of the optical matrix.

[3] *Sufficient* conditions on ρ_{ij} for $\boldsymbol{\ell}$ to be a multiple WAND are not in full generality known, but it has been shown [5, 6] that $\rho_{ij}=0$ (Kundt class) and $\rho_{ij} \propto \delta_{ij}$ (Robinson-Trautman class) are examples of such sufficient conditions.

The same form of $\boldsymbol{\rho}$ also applies to type III Ricci flat spacetimes [7] that either

- (i) are five-dimensional,
- (ii) satisfy a certain genericity condition [7],
- (iii) have a *non-twisting* multiple WAND (with vanishing rotation, i.e., $a = 0$).

Generalization of the above type N and III results from the Ricci-flat case to the Einstein case is straightforward, see [8]. Note that ρ_{ij} of the form (2) is shearfree for $d = 4$ but not for $d > 4$ (for $b \neq 0$).

1.1.2 Kerr-Schild spacetimes

It can be shown [9, 10] that for Kerr-Schild (KS) spacetimes [4]

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + H k_\mu k_\nu, \tag{3}$$

with vanishing $T_{00} \equiv T_{ab}k^a k^b$ component of the energy–momentum tensor the KS vector k is a geodetic multiple WAND and the optical matrix can be put into a block diagonal form

$$\boldsymbol{\rho} = \alpha \mathrm{diag}\left(1, \dots 1, \frac{1}{1+\alpha^2 b_1^2}\begin{bmatrix} 1 & -\alpha b_1 \\ \alpha b_1 & 1 \end{bmatrix}, \dots, \frac{1}{1+\alpha^2 b_\nu^2}\begin{bmatrix} 1 & -\alpha b_\nu \\ \alpha b_\nu & 1 \end{bmatrix}, 0, \dots, 0\right). \tag{4}$$

1.1.3 Asymptotically flat type II spacetimes

Similar result as above holds also for asymptotically flat type II spacetimes [11].

It can be shown that in this case $\boldsymbol{\rho}$ obeys the *optical constraint*

$$\rho_{ik}\rho_{jk} \propto \rho_{(ij)}, \tag{5}$$

which in fact holds if and only if $\boldsymbol{\rho}$ can be put in the canonical form (4) by appropriately choosing the frame. Note that the optical constraint also holds in the type N and III cases discussed above. In fact it turns out that in arbitrary dimension such form of $\boldsymbol{\rho}$ is "prefered" and holds for generic algebraically special spacetimes (see [12] for more precise formulation). Explicit examples of Einstein spacetimes with all multiple WANDs violating the optical constraint are however also known [2].

[4] $\bar{g}_{\mu\nu}$ is a metric of constant curvature and KS vector k is null with respect to $\bar{g}_{\mu\nu}$ and thus also with respect to $g_{\mu\nu}$.

2 GS theorem in five dimensions

Let us now summarize main results of [12], where the necessary conditions for $\boldsymbol{\ell}$ to be a multiple WAND in a five-dimensional algebraically special Einstein spacetime have been found:

Theorem 1. *In a five-dimensional algebraically special Einstein spacetime that is not conformally flat, there exists a geodetic multiple WAND $\boldsymbol{\ell}$ and its optical matrix can be put in one of the forms*

$$i) \quad b\begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1+a^2 \end{pmatrix}, \tag{6}$$

$$ii) \quad b\begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

$$iii) \quad b\begin{pmatrix} 1 & a & 0 \\ -a & -a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

If the spacetime is of type III or type N then the form must be ii).

Note that for $b \neq 0$ matrices i), ii), iii) have rank 3, 2, 1, respectively, and for $b = 0$ it is the Kundt spacetime. Only the case iii) with $a \neq 0 \neq b$ does not satisfy the optical constraint (5). However, it has been proven recently that this case cannot occur for genuine type II spacetimes [13] (see also [14]) and for type D one can show [12]

Proposition 2. *A five-dimensional type D Einstein spacetime admits a geodetic multiple WAND violating the optical constraint if, and only if, it admits a non-geodetic multiple WAND.*

Note that all 5d Einstein spacetimes admitting a non-geodetic multiple WAND are explicitly known [4]. It turns out that these spacetimes always admit another geodetic multiple WAND satisfying optical constraint and one can thus conclude

Proposition 3. *A five-dimensional algebraically special Einstein spacetime always admits a geodetic multiple WAND obeying the optical constraint.*

Due to Theorem 1 the number of independent components of the optical matrix $\boldsymbol{\rho}$ is reduced from 6 to 2 free parameters. This will lead to a considerable simplification of the GHP equations [8] and hopefully also to a discovery of new higher dimensional algebraically special Einstein spacetimes.

Let us conclude with some known examples of Einstein spacetimes belonging to the cases i) - iii) of the Theorem 1 [12]

- case i)
  $a \neq 0$: the Myers-Perry [15] black hole solution (cf. [16]) and in fact all non-degenerate (i.e. $\det\boldsymbol{\rho} \neq 0$) Einstein Kerr-Schild metrics with Minkowski or (A)dS

background [9, 10], 5d Kaluza-Klein bubble obtained by analytic continuation of a singly spinning Myers-Perry solution [17, 12];
$a = 0$: this case corresponds to the Robinson-Trautman class [6] and in five dimensions it reduces to the Schwarzschild-Tangherlini metric (possibly with a cosmological constant).

- case ii)
  $a \neq 0$: product of a 4d Ricci-flat algebraically special twisting solution with a flat 5th direction, e.g., the Kerr black string (i.e. the product of the 4d Kerr solution with a flat direction), or more generally warped product of a 4d algebraically special Einstein spacetime with a 5th direction (with non-vanishing cosmological constant);
  $a = 0$: a direct or warped product of any 4d Einstein type II Robinson-Trautman metric [18], e.g. the Schwarzschild black string solution.

- case iii)
  Spacetimes belonging to this case are of type D ([13], see also [14]) and admit a non-geodetic multiple WAND. All such metrics were determined in [4] – the direct products $dS_3 \times S^2$ and $AdS_3 \times H^2$ or the analytical continuation of the 5d Schwarzschild solution [19] (generalized to include a cosmological constant and planar or hyperbolic symmetry)
  $a \neq 0$: a specific twisting geodetic multiple WAND, e.g., in $dS_3 \times S^2$, see [12];
  $a = 0$: a non-twisting, expanding and shearing geodetic multiple WAND in $dS_3 \times S^2$ or in the Kaluza-Klein bubble solution, see [12].

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