Electric and Magnetic Weyl Tensors in Higher Dimensions

S. Hervik, M. Ortaggio, and L. Wylleman

Abstract Recent results on purely electric (PE) or magnetic (PM) spacetimes in n dimensions are summarized. These include: Weyl types; diagonalizability; conditions under which direct (or warped) products are PE/PM.

1 Definition and general properties

The standard decomposition of the Maxwell tensor F_{ab} into its electric and magnetic parts $\mathbf{E}$ and $\mathbf{B}$ with respect to (wrt) an observer (i.e., a unit time-like vector u) can be extended to any tensor in an n-dimensional spacetime [1, 2, 3]. Here we summarize the results of [3] about the Weyl tensor, and the connection with the null alignment classification [4, 5].

Consider the u-orthogonal projector $h_{ab} = g_{ab} + u_a u_b$. The "electric" and "magnetic" parts of C_{abcd} can be defined, respectively, as [3]

$$(C_+)^{ab}{}_{cd} = h^{ae}h^{bf}h_c{}^g h_d{}^h C_{efgh} + 4u^{[a}u_{[c}C^{b]e}{}_{d]f}u_e u^f, \quad (1)$$

$$(C_-)^{ab}{}_{cd} = 2h^{ae}h^{bf}C_{efk[c}u_{d]}u^k + 2u_k u^{[a}C^{b]kef}h_{ce}h_{df}. \quad (2)$$

S. Hervik
Faculty of Science and Technology, University of Stavanger, N-4036 Stavanger, Norway
e-mail: sigbjorn.hervik@uis.no

M. Ortaggio
Institute of Mathematics, Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Prague 1, Czech Republic
e-mail: ortaggio@math.cas.cz

L. Wylleman
Faculty of Applied Sciences TW16, Ghent University, Galglaan 2, 9000 Gent, Belgium
e-mail: lode.wylleman@ugent.be

These extend the well-known 4D definitions [6, 7]. In any orthonormal frame adapted to u the electric [magnetic] part accounts for the Weyl components with an even [odd] number of indices u. *At a spacetime point (or region) the Weyl tensor is called "purely electric [magnetic]" (from now on, PE [PM]) wrt u if* $C_- = 0$ $[C_+ = 0]$. The corresponding spacetime is also called PE [PM]. Several conditions on PE/PM Weyl tensors follow.

Proposition 1 (Bel-Debever-like criteria [3]). *A Weyl tensor* C_{abcd} *is:*
(i) PE wrt u iff $u_a g^{ab} C_{bc[de}u_{f]} = 0$*; (ii) PM wrt u iff* $u_{[a}C_{bc][de}u_{f]} = 0$.

Proposition 2 (Eigenvalues [3]). *A PE [PM] Weyl operator*[1] *is diagonalizable, and possesses only real [purely imaginary] eigenvalues. Moreover, a PM Weyl operator has at least* $\frac{(n-1)(n-4)}{2}$ *zero eigenvalues.*

Proposition 3 (Algebraic type [3]). *A Weyl tensor which is PE/PM wrt a certain u can only be of type G,* I_i*, D or O. In the type* I_i *and D cases, the second null direction of the timelike plane spanned by u and any WAND is also a WAND (with the same multiplicity). Furthermore, a type D Weyl tensor is PE iff it is type* $D(d)$*, and PM iff it is type* $D(abc)$.

Proposition 4 (Uniqueness of u [3]). *A PE [PM] Weyl tensor is PE [PM] wrt: (i) a unique u (up to sign) in the type* I_i *and G cases; (ii) any u belonging to the space spanned by all double WANDs (and only wrt such us) in the type D case (noting also that if there are more than two double WANDs the Weyl tensor is necessarily PE (type* $D(d)$*)* [10]).

2 PE spacetimes

Proposition 5 ([3]). *All spacetimes admitting a shearfree, twistfree, unit timelike vector field u are PE wrt u. In coordinates such that* $u = V^{-1}\partial_t$*, the line-element reads*

$$\mathrm{d}s^2 = -V(t,x)^2\mathrm{d}t^2 + P(t,x)^2\xi_{\alpha\beta}(x)\mathrm{d}x^\alpha\mathrm{d}x^\beta\,. \tag{3}$$

The above metrics include, in particular, direct, warped and doubly warped products with a one-dimensional timelike factor, and thus all *static* spacetimes (see also [11]). For a warped spacetime (M,g) with $M = M^{(n_1)}\times M^{(n_2)}$, one has $g = e^{2(f_1+f_2)}\left(g^{(n_1)}\oplus g^{(n_2)}\right)$, where $g^{(n_i)}$ is a metric on the factor space $M^{(n_i)}$ $(i = 1,2)$ and f_i are functions on $M^{(n_i)}$ ($M^{(n_i)}$ has dimension n_i, $n = n_1 + n_2$, and $M^{(n_1)}$ is Lorentzian).

Proposition 6 (Warps with $n_1 = 2$ [11, 3]). *A (doubly) warped spacetime with* $n_1 = 2$ *is either type O, or type D(d) and PE wrt any u living in* $M^{(n_1)}$*; the uplifts of the*

[1] In the sense of the Weyl operator approach of [8] (see also [9]).

null directions of the tangent space to $(M^{(n_1)}, g^{(n_1)})$ *are double WANDs of* (M,g)*. If* $(M^{(n_2)}, g^{(n_2)})$ *is Einstein the type specializes to D(bd), and if it is of constant curvature to D(bcd).*

In particular, all spherically, hyperbolically or plane symmetric spacetimes belong to the latter special case.

Proposition 7 (Warps with $n_1 = 3$ [11, 3]). *A (doubly) warped spacetime with* $(M^{(n_1)}, g^{(n_1)})$ *Einstein and* $n_1 = 3$ *is of type D(d) or O. The uplift of* any *null direction of the tangent space to* $(M^{(n_1)}, g^{(n_1)})$ *is a double WAND of* (M,g)*, which is PE wrt any u living in* $M^{(n_1)}$*.*

Proposition 8 (Warps with $n_1 > 3$ [11, 3]). *In a (doubly) warped spacetime*

(i) if $(M^{(n_1)}, g^{(n_1)})$ *is an Einstein spacetime of type D,* (M,g) *can be only of type D (or O) and the uplift of a double WAND of* $(M^{(n_1)}, g^{(n_1)})$ *is a double WAND of* (M,g)

(ii) if $(M^{(n_1)}, g^{(n_1)})$ *is of constant curvature,* (M,g) *is of type D(d) (or O) and the uplifts of* any *null direction of the tangent space to* $(M^{(n_1)}, g^{(n_1)})$ *is a double WAND of* (M,g)*;* (M,g) *is PE wrt any u living in* $M^{(n_1)}$*.*

Proposition 9 (PE direct products [3]). *A direct product spacetime* $M^{(n)} = M^{(n_1)} \times M^{(n_2)}$ *is PE wrt a u that lives in* $M^{(n_1)}$ *iff u is an eigenvector of* $R^{(n_1)}_{ab}$*, and* $M^{(n_1)}$ *is PE wrt u. (u is then also an eigenvector of the Ricci tensor* R_{ab} *of* $M^{(n)}$*, i.e.,* $R_{ui} = 0$*.)*

A conformal transformation (e.g., to a (doubly) warped space) will not, of course, affect the above conclusions about the Weyl tensor. There exist also direct products which are PE wrt a vector u *not* living in $M^{(n_1)}$ [3].

Also the presence of certain (Weyl) isotropies (e.g., $SO(n-2)$ for $n > 4$) implies that the spacetime is PE, see [8, 3] for details and examples.

3 PM spacetimes

Proposition 10 (PM direct products [3]). *A direct product spacetime* $M^{(n)} = M^{(n_1)} \times M^{(n_2)}$ *is PM wrt a u that lives in* $M^{(n_1)}$ *iff all the following conditions hold (where* $R_{(n_i)}$ *is the Ricci scalar of* $M^{(n_i)}$*):*

i) $M^{(n_1)}$ *is PM wrt u and has a Ricci tensor of the form* $R^{(n_1)}_{ab} = \frac{R_{(n_1)}}{n_1} g^{(n_1)}_{ab} + u_{(a}q_{b)}$ *(with* $u^a q_a = 0$*)*

ii) $M^{(n_2)}$ *is of constant curvature and* $n_2(n_2-1)R_{(n_1)} + n_1(n_1-1)R_{(n_2)} = 0$*.*

Further, $M^{(n)}$ *is PM Einstein iff* $M^{(n_1)}$ *is PM Ricci-flat and* $M^{(n_2)}$ *is flat.*

See [3] for explicit (non-Einstein) examples. However, in general PM spacetimes are most elusive. For example,

Proposition 11 ([3]). *PM Einstein spacetimes of type D do not exist.*

In [3] also several results for PE/PM Ricci and Riemann tensors have been worked out, along with corresponding examples. In general, we observe that PE/PM tensors provide examples of *minimal tensors* [12]. Thanks to the *alignment theorem* [13], the latter are of special interest since they are precisely the *tensors characterized by their invariants* [13] (cf. also [3]). This in turn sheds new light on the classification of the Weyl tensor [5], providing a further invariant characterization that distinguishes the (minimal) types G/I/D from the (non-minimal) types II/III/N.

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