Stability of marginally outer trapped surfaces and geometric inequalities

Marc Mars

Abstract Marginally outer trapped surfaces (MOTS) admit a notion of stability that in many respects generalizes a similar notion for minimal hypersurfaces. Stable MOTS play an interesting role in a number of geometric inequalities involving physical parameters such as area, mass, charge or, in the axially symmetric case, angular momentum. Some of those inequalities are global in nature while others are local, with interesting relationships between them. In this lecture the notion of stable MOTS will be reviewed and some of the geometric inequalities involving stable MOTS will be described.

1 Introduction

Geometric inequalities play a fundamental role in gravitation because they provide information on the relationship between physically relevant quantities in a robust way, independently of the details of the particular spacetime under consideration. It is often the case that not even field equations are necessary for the validity of such inequalities and that only energy conditions are required, which make their range of validity very broad and transverse to several theories of gravity. One of the most fundamental geometric inequalities in gravitation is the Positive Mass Theorem which, as is well-known establishes that the ADM mass is non-negative for any asymptotically flat spacetime satisfying the dominant energy condition (DEC), i.e. such that the Einstein tensor contracted with any pair of future directed causal vectors gives a non-positive quantity. Another very important geometric inequality, which so far has been proved only in special circumstances is the Penrose inequality [1]. In the asymptotically flat case, this inequality conjectures that the total ADM mass M satisfies the inequality

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$$|S_{\min}| \leq 16\pi M^2.$$

where $|S_{\min}|$ is the minimal area needed to enclose a given weakly future rapped surface S, i.e. a closed, spacelike, codimension-two surface with future directed, causal mean curvature vector. The notion of "enclosing" also needs a definition, see e.g. [2] for details. The Penrose inequality is a strengthening of the positive mass theorem when there are trapped surfaces present. In turn, the Penrose inequality can be strengthened when the spacetime is charged and the total charge Q cannot be radiated away (e.g. in electrovacuum, or when all matter present in the spacetime is electrically neutral). In this case, the conjectured inequality becomes [3]

$$|S_{\min}| \leq 8\pi \left(M^2 - \frac{Q^2}{2} + \sqrt{M^2 \left(M^2 - Q^2\right)} \right).$$

It is clear that, in order for this inequality to even make sense, it is necessary that the total ADM mass satisfies the bound $M \geq |Q|$. This inequality was proved in [4] (see also [5] for a mathematically complete argument) and provides a direct strengthening of the positive mass theorem in charged spacetimes, irrespectively of whether a weakly future trapped surface is present in the spacetime or not.

Another global charge in asymptotically flat spacetimes is the ADM angular momentum J. In general, angular momentum can be radiated away by gravitational waves, so in general no strengthening of the Penrose inequality involving angular momentum should be expected (this is because the physical argument leading to the Penrose inequality involves, on the one hand, the weak cosmic censorship hypothesis and, on the other, the asymptotic values of mass, charge and angular momentum of the black hole that forms during the collapse). There is one interesting case, however, when angular momentum cannot be radiated away, namely when the spacetime is axially symmetric. Thus, in this case (and assuming again that electric charge is either absent or cannot be radiated away) the Penrose inequality can be strengthened to (see [6, 7] for a discussion)

$$|S_{\min}| \leq 8\pi \left(M^2 - \frac{Q^2}{2} + \sqrt{M^4 - Q^2 M^2 - J^2} \right) := F_{Q,J}(M), \tag{1}$$

where the last equality defines a function of M for each choice of Q and J. As before, in order for this inequality to make sense, it becomes necessary that $M^4 - Q^2M^2 - J^2 \geq 0$ or, equivalently,

$$M^2 \geq \frac{1}{2}(Q^2 + \sqrt{Q^4 + 4J^2}).$$

This is a strengthening of the positive mass theorem that should be valid for any asymptotically flat, axially symmetric spacetime with conserved total electric charge. This result has been proved whenever the spacetime admits a maximal (i.e. with second fundamental form of vanishing trace), axially symmetric slice with two asymptotically flat ends, first in vacuum [8, 9] and then in electrovacuum [10, 11]. The assumption of having more than one asymptotically flat end is made because, in electrovacuum, the total electric charge and the total angular momentum can only be non-zero in the presence of non-trivial topology. In the case of vacuum with an arbitrary number of asymptotically flat ends, the inequality is not yet settled but a closely related inequality has been proved in [12], which possibly reduces to the previous one depending on the values of a certain functional defined on stationary and axially symmetric asymptotically flat spacetimes (see the review [13] for many more details).

All the geometric inequalities discussed so far are global in nature because they involve the total ADM mass, and the charge and angular momentum are also global quantities defined at infinity. Even the area term in the Penrose inequality is global quantity because of the need of taking the minimal area enclosure of the weakly future trapped surface. It is straightforward to check that $F_{Q,J}$ is an increasing function of M. Its minimum value is $4\pi\sqrt{Q^4+4J^2}$. It therefore follows that, whenever the minimal area enclosure $|S_{\min}|$ satisfies $|S_{\min}| \leq 4\pi\sqrt{Q^4+4J^2}$ then the Penrose inequality (1) is satisfied automatically. It follows that this inequality is non-trivial only if

$$|S_{\min}| \geq 4\pi\sqrt{Q^4+4J^2}.$$

In particular, when the total charge is non-conserved or vanishes identically, this inequality becomes

$$|S_{\min}| \geq 8\pi|J|$$

and when the total angular momentum is non-conserved, the inequality reads

$$|S_{\min}| \geq 4\pi Q^2.$$

These inequalities are still of global nature, but now it makes sense to try and see whether a local version of them is still valid. Indeed, one may think of replacing the minimal area enclosure $|S_{\min}|$ by the area $|S|$ of the weakly future trapped surface itself. Moreover, the notion of total charge enclosed by a closed, orientable surface makes sense, and in the axially symmetric case, the presence of a Killing vector allows one to define the Komar angular momentum of any closed, orientable surface. Moreover, in electrovacuum there is a modification [14] of the Komar definition of angular momentum involving the electromagnetic field which provides a conserved quantity in the sense that it gives an object depending only on the homology class of the surface under consideration.

There has been very interesting and remarkable progress in recent years towards the proof of local inequalities of this type on certain surfaces. They were first found in the case of stationary and axially symmetric black hole horizons admitting arbitrary matter outside the horizon but such that a neighbourhood of the horizon itself is vacuum or electrovacuum. The first result along those lines was for the degenerate case (i.e. when the surface gravity of the horizon vanishes) [15]. The non-degenerate case for vacuum horizons was dealt with in [16]. Finally, the inequality in the charged, rotating case and for electrovacuum horizons was solved in [17].

These results had important implications in the problem of non-existence of stationary and axially symmetric two-black hole configurations [18, 19, 20, 21] (see also G. Neugebauer's contribution to this volume).

Remarkably, a purely local version of this inequality where only properties of suitable spacelike two-surfaces are used has also been obtained recently. The first case, proved by Dain and Reiris [22] involved stable minimal, axially symmetric surfaces embedded in maximal, axially symmetric hypersurfaces in a vacuum spacetime. In this setting, the universal inequality $|S| \geq 8\pi|J|$ was proved, where J is the Komar angular momentum. This inequality was then extended [23] to arbitrary, axially symmetric, stable marginally outer trapped surfaces (defined below) embedded in a spacetime with arbitrary matter contents as long as the dominant energy condition is satisfied. The case with electric charge (and no angular momentum so that no need to restrict oneself to axially symmetric situations) was analyzed in [24] where the inequality $|S| \geq 4\pi Q^2$ was proved for suitable surfaces. The case involving both charge and angular momentum has been proved recently in [25].

The key underlying property of the local versions of the inequality is the notion of stability, both for minimal hypersurfaces and for marginally outer trapped surfaces. The aim of this lecture is to review the notion of stability for marginally outer trapped surfaces and discuss some of its consequences. Then, I will present in more detail the various inequalities and explain how does stability enter into the arguments. The final aim will be to relate the black hole-type inequalities to the purely local inequalities by summarizing recent results [26] on the stability properties of Killing horizons.

2 Basics on the geometry of spacelike surfaces.

Our framework will be a four-dimensional spacetime (M,g), which we will take to be oriented and time-oriented. Scalar product with the spacetime metric will be denoted by $\langle\, ,\rangle$ and S will refer to a closed (i.e. compact without boundary), spacelike, two-dimensional, orientable, connected, embedded surface in (M,g) (simply *surface* from now on). The normal space to S at any of its points is a Lorentzian vector space. The collection of all normal spaces is a vector bundle over S that admits two global, smooth, nowhere zero cross-sections $\{\ell,k\}$ satisfying $\langle \ell,\ell\rangle = 0$, $\langle k,k\rangle = 0$ and $\langle \ell,k\rangle = -1$. We always take ℓ (and hence k) to be future directed. These sections are defined uniquely up to the usual boost freedom $\ell \to F\ell$, $k \to F^{-1}k$, where F is a smooth, positive scalar function on S. The (positive definite) induced metric on S will be denoted by h and the corresponding covariant derivative by D. The null extrinsic curvatures are defined, as usual, by $\chi_{\ell AB} \equiv \langle e_A, \nabla_{e_B}\ell\rangle$, $\chi_{kAB} \equiv \langle e_A, \nabla_{e_B}k\rangle$, where $\{e_A\}$ is a basis of the tangent space of S. The null expansions are the traces of these tensors, i.e. $\theta_\ell = \mathrm{tr}_h(\chi_\ell)$, $\theta_k = \mathrm{tr}_h(\chi_k)$. A relevant geometric object in the following is the mean curvature vector, defined in terms of the null expansions by $H = -\theta_\ell k - \theta_k \ell$. It is well-known (and straightforward) that this vector is independent of the choice of null basis $\{\ell, k\}$. Finally, the normal bundle admits a canonical connection with connection one-form given by $s_A \equiv -\langle k, \nabla_{e_A} \ell \rangle$ in the basis $\{\ell, k\}$.

The mean curvature is fundamental, among other things, because it contains full information on how the area of the surface changes to first order under general deformations. Denoting by $\delta_\xi |S|$ the first order variation of area along a deformation vector ξ, the following identity holds

$$\delta_\xi |S| = \int_S \langle H, \xi \rangle \eta_h,$$

where η_h is the metric volume form of (S, h). Hence, when H is future causal then the area of S does not increase for any future causal deformation ξ. This is generally taken as a clear signal of the presence of a strong gravitational field. A surface S with this property is called weakly future trapped.

3 Marginally outer trapped surfaces and stability

As just mentioned, the future causal character for the mean curvature is sufficient to signal the presence of a strong gravitational field. However, if the surface admits a well-defined notion of exterior (for instance when S is contained in a spacelike, asymptotically flat hypersurface and separates this hypersurface into an asymptotically flat exterior and a compact domain), then a non-increase of area along the exterior future null cone may also be taken as convincing indication that the gravitation field on the surface is strong. Such surfaces are defined by the property that $\theta_\ell \leq 0$, where ℓ is the future null normal pointing into the exterior domain.

The borderline case is given by surfaces satisfying the equality case $\theta_\ell = 0$. It turns out that studying this borderline case is interesting even when there is no clearly distinguished notion of exterior. This leads to the following standard definition:

Definition 1. A marginally outer trapped surface (MOTS) is a surface which satisfies either $\theta_\ell = 0$ or $\theta_k = 0$ everywhere (after renaming we always take $\theta_\ell = 0$).

From the general expression for the first variation of area it follows that MOTS are stationary points for the area functional with respect to arbitrary variations tangent to ℓ. Stationary points of any functional call immediately for analyzing their behaviour under second order variations. In this case, the result is a direct consequence of the Raychauduri equation and gives

$$\delta^2_{\psi\ell} |S| = -\int_S \psi^2 \left(G(\ell, \ell) + |\chi_\ell|^2 \right) \eta_S,$$

where G is the Einstein tensor of the spacetime and $|\cdot|$ is the norm of tensors on S with the metric h. In contrast to the case of minimal hypersurfaces in Riemannian ambient manifolds, the second order variation does not define a differential operator acting on ψ. Moreover, it is always non-positive definite, provided the spacetime satisfies the null energy condition NEC ($G(\ell', \ell') \geq 0$ for any null vector ℓ'). Thus, the second variation of area along the direction ℓ for MOTS does not provide with any useful notion of stability. A suitable alternative is to use first variations of θ_ℓ (which vanishes for a MOTS) along arbitrary normal directions not tangent to ℓ. To be more precise, select a normal direction to S nowhere tangent to ℓ. This defines uniquely a vector v satisfying $\langle \ell, v \rangle = 1$. Consequently, it can be decomposed uniquely as $v = -k + V\ell$ where V is a scalar function on S. In other words, we parametrize the collection of directions orthogonal to S and nowhere tangent to ℓ by a real function $V : S \to \mathbb{R}$ via the generator v defined above.

Note that V may a priory have any value. The sign of V at any given point $p \in S$ is directly tied to the causal character of the normal direction selected at p. If $V > 0$ then this direction is spacelike, if $V = 0$ the direction is null and if $V < 0$ the direction is timelike. It is also convenient to define the vector $v^\star := k + V\ell$, which defines a second normal direction. This direction is linearly independent to the previous one except when v is null, in which case they coincide. In any other case the causal character of $v^\star$ is always opposite (in the sense of spacelike vs. timelike) to the causal character of v. In particular, $v^\star$ is future causal if and only if v is "achronal" (i.e. spacelike or null or, equivalently, $V \geq 0$).

Given a fixed direction defined by v we can perform variations restricted to this direction. Any such variation is defined by a vector $\xi = \psi v$, where $\psi : S \to \mathbb{R}$. The first order variation of θ_ℓ along ψv was first computed by Newman [27] (the calculation was performed assuming implicitly that $\psi \neq 0$ everywhere, but the result is generally valid, c.f. [28]). The resulting expression gives a differential operator L_v acting ψ via the definition $L_v(\psi) \equiv \delta_{\psi v} \theta_\ell$. Its explicit expression is

$$L_v(\psi) = -\Delta_h \psi + 2 s_A D^A \psi + \left(\frac{R(h)}{2} - G(\ell, v^\star) - V |\chi_\ell|^2 - s_A s^A + D_A s^A \right) \psi, \quad (2)$$

where Δ_h is the Laplacian of (S, h) and $R(h)$ is the curvature scalar of the metric h.

As discussed in detail in [28] this operator is elliptic and, in general, not self-adjoint. However, any elliptic operator on a compact manifold (or on a bounded domain with Dirichlet boundary conditions) admits a principal eigenvalue λ_v. This is a real eigenvalue (i.e. such that there exists a real function ϕ_v satisfying $L_v(\phi_v) = \lambda_v \phi_v$) with the property that any other eigenvalue (which will be complex in general) satisfies $\mathrm{Re}(\lambda) > \lambda_v$. Moreover, as in the self-adjoint case, the principal eigenfunction ϕ_v does not change sign and hence can be taken to be positive everywhere. For self-adjoint operators, the principal eigenvalue admits a characterization in terms of the so-called Rayleigh-Ritz quotient which is very useful, firstly because it provides upper bounds for the principal eigenvalue and secondly, and even more importantly, because it gives lower bounds for certain integral functionals acting on arbitrary functions. This latter property is the key for translating sign conditions on the principal eigenvalues into useful analytic inequalities that, in turn, can be used to derive geometric properties of the surface. This is crucial, e.g., for studying stable minimal hypersurfaces in Riemannian manifolds. Such surfaces have the property that their own stability operator, which is now self-adjoint, has a non-negative principal eigenvalue.

The Rayleigh-Ritz characterization is no longer true for non self-adjoint operators. Given its importance, it is reasonable to ask whether there exists any analogue to this characterization valid for any elliptic operator. Donsker and Varadhan [29] found a number of characterizations of the principal eigenvalue of the min-max type, i.e. involving a minimization of a certain class of suprema. Such characterizations are, in general, difficult to work with. In [28] one of these characterizations was elaborated further and a Rayleigh-Ritz type characterization for the principal eigenvalue of any elliptic second order operator was found. In order to describe it, recall that any one-form in a compact Riemannian manifold without boundary can be decomposed as the sum of the differential of a function f plus a divergence-free one-form. Such decomposition is usually called Helmholz decomposition in the physics literature. Recall that this decomposition is unique up to an additive constant in f provided the manifold is connected.

In the case of the stability operator L_v, the characterization of λ_v obtained in [28] reads as follows.

Proposition 1. *Let L_v be the stability operator (2). Decompose the normal connection one-form s_A according to the Helmoltz decomposition as $s_A = D_A f + z_A$, where z_A is divergence-free. Then the principal eigenvalue of L_v is given by*

$$\lambda_v = \inf_{u>0} \frac{\int_S \left(|Du|^2 + \left(\frac{R(h)}{2} - G(\ell, v^\star) - V|\chi_\ell|^2 \right) u^2 - |d\omega_u + z|^2 u^2 \right) \eta_h}{\int_S u^2 \eta_h} \tag{3}$$

where ω_u is any solution of

$$-\Delta_h \omega_u - \frac{2}{u} D_A \omega_u D^A u = \frac{2}{u} z_A D^A u. \tag{4}$$

It is straightforward to check that, given any positive function u on S, the partial differential equation (4) always admits a solution, which is unique up to an additive constant.

An immediate consequence of this result is the so-called "symmetrized inequality along v" proved by Galloway and Schoen [30] using explicit estimates,

$$\int_S \left(|Du|^2 + \frac{1}{2} R(h) u^2 \right) \eta_h \geq \int_S \left(\lambda_v + G(\ell, v^\star) + V|\chi_\ell|^2 \right) u^2 \eta_h. \tag{5}$$

Indeed, by dropping the term $|d\omega_u + z|^2 u^2$ in (3) the right-hand side is never decreased, and hence so does infimum, which leads immediately to (5).

The stability operator allows us to define a notion of stability for MOTS [28], which generalizes a similar notion for minimal hypersurfaces.

Definition 2. A MOTS S is **stable along** v if and only if the principal eigenvalue λ_v of the stability operator satisfies $\lambda_v \geq 0$. S is **strictly stable along** v if and only if $\lambda_v > 0$.

One of the consequences of this definition is given by the following lemma, which links the stability of a MOTS with the existence of suitable variations which increase the value of the outer null expansion.

Lemma 1. *A MOTS S is stable along v if there exists an outward variation ψv ($\psi \geq 0$, $\psi \not\equiv 0$) such that $\delta_{\psi v}\theta_\ell \geq 0$. S is strictly stable if, in addition, $\delta_{\psi v}\theta_\ell > 0$.*

Although, by construction, there is a stability operator (and a notion of stability) for each direction v, the dependence of L_v on v is very simple, namely $L_v = L_{-k} - WV$, where W is defined as $W := (G(\ell,\ell) + |\chi_\ell|^2)$. Note that under the NEC $W \geq 0$ and hence stability improves when v is tilted away from ℓ (i.e. when V is made larger at every point). In fact, when $W \not\equiv 0$, there always exists a direction sufficiently tilted away from ℓ for which the principal eigenvalue is positive, and hence the MOTS is strictly stable along this direction. However, it turns out that the stability of the MOTS along a direction v gives, in general, useful information only when the direction v is achronal. This is because, the terms $G(\ell, v^\star)$ and $V|\chi_\ell|^2$ are non-negative (under the NEC) only when $V \geq 0$, i.e. when v is spacelike or null at every point. This leads to the following definition, spelled out in [23] and closely related to the notion of future outer trapping horizon defined by Hayward [31].

Definition 3. A MOTS S is **spacetime stable** if it is stable along an achronal direction v.

It is now straightforward to check that, under the NEC, S is spacetime stable if and only if it is stable along $-k$. Since the null direction $-k$ is privileged as a transverse direction for the MOTS, we will write λ_{-k} simply as λ_-.

4 Area-charge-topology inequalities for MOTS

As discussed in the Introduction, a class of local inequalities exists for suitable surfaces relating the area and the total charge enclosed by the surface. If the surface has non-trivial topology, its genus (recall that our surfaces are two-dimensional, connected and orientable, so that their topology is uniquely determined by their genus) also enters into the inequality. The basic assumption made in this context is that the energy-momentum contents of the spacetime splits into an electromagnetic field and the rest of matter in such a way that this rest satisfies the dominant energy condition. Allowing also a cosmological constant Λ, the Einstein tensor takes the form (assuming we are in General Relativity) $G_{\mu\nu} + \Lambda g_{\mu\nu} = T^{\rm EM}_{\mu\nu} + T^{\rm mat}_{\mu\nu}$, where $T^{\rm EM}_{\mu\nu} = 2(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta})$ is the energy-momentum tensor of the electromagnetic field $F_{\mu\nu}$ and $T^{\rm mat}$ is arbitrary except for the condition of satisfying DEC. The electric and magnetic charges on S are defined as

$$Q_E = \frac{1}{4\pi}\int_S F(\ell,k)\eta_h, \qquad Q_M = \frac{1}{4\pi}\int_S F^{\star}(\ell,k)\eta_h,$$

where $F^{\star}$ is the Hodge dual of F. We now assume that S is a MOTS. If we insert $u = 1$ in the symmetrized inequality (5) along $-k$ (i.e. with $V = 0$) it follows

$$4\pi(1-g) \geq (\lambda + \Lambda)\,|S| + \int_S (F(\ell,k))^2 + (F^{\star}(\ell,k))^2 \eta_h \tag{6}$$

Now, following [24], the Hölder inequality implies the inequality $\int_S f^2 \geq |S|^{-1}(\int_s f)^2$ valid for any L^2 function f. Thus, (6) implies

$$4\pi(1-g) \geq (\lambda + \Lambda)|S| + 16\pi^2|S|^{-1}(Q_E^2 + Q_M^2).$$

This inequality including the cosmological constant, arbitrary topology of S and the principal eigenvalue has been obtained in [32], where a number of consequences have also been derived. Two immediate consequences are the following:

Assume that S is spacetime stable. Since, under the assumptions above, the spacetime satisfies the NEC (irrespectively of the sign of Λ) it follows that S is stable along the direction $-k$, i.e. $\lambda_- \geq 0$. If we assume further that $\Lambda \geq 0$, then it follows immediately that S must be of spherical or toroidal topology. This recovers a well-known theorem on the topology of MOTS due to Hawking [33]. This theorem has been generalized to higher dimensions in [30] where it was shown that the Yamabe type of any stable MOTS must be non-negative. The case of vanishing Yamabe type (i.e. toroidal topology in the case of four dimensions) has been shown to be very rigid in [34] and to be excluded when the MOTS satisfies suitable barrier properties.

If S is spacetime stable and there is a negative cosmological constant, then (7) implies an upper bound on the genus of S, namely $g \leq 1 - \frac{|S|\Lambda}{4\pi}$. This bound was first obtained in [35].

For the area-charge inequality, (7) immediately implies that, as long as S is spacetime stable and $\Lambda \geq 0$, then $|S| \geq 4\pi(Q_E^2 + Q_M^2)$, as first proved in [24]. This inequality, in turn, is a generalization of a previous result by Gibbons which establishes this inequality in the case of minimal surfaces embedded a time-symmetric slice [36].

5 Axially symmetric MOTS and angular momentum

We next discuss local geometric inequalities involving angular momentum. As mentioned in the Introduction, this case requires the surfaces to be axially symmetric in order to have a proper definition of angular momentum. In principle, one may think that this entails restricting oneself to axially symmetric spacetimes. In fact, fewer requirements are needed and the following definition, essentially put forward in [23], is sufficient for the purpose of writing down and proving the desired inequalities:

Definition 4. A MOTS S is axially symmetric if there exists a vector η tangent to S satisfying

1. $\mathscr{L}_\eta h = 0$.
2. $\mathscr{L}_\eta s = 0$, for some choice of basis $\{\ell, k\}$.
3. η commutes with the stability operator L_v for some choice of v.

It is clear that if the spacetime (M,g) admits an axial Killing vector η and η is tangent to S, then the S is also axially symmetric according to this definition.

The angular momentum J of an axially symmetric MOTS is then defined by

$$J(S) := \frac{1}{8\pi}\int_S s(\eta)\eta_S.$$

It is straightforward to check that this definition is independent of the choice of basis $\{\ell, k\}$. Nevertheless, for some expressions below, it is necessary to restrict the basis to satisfy point 2. in definition 4. We will do so from now on.

The following theorem due to Jaramillo, Reiris and Dain [23] establishes a remarkable inequality involving the area and the angular momentum of an axially symmetric MOTS. The only requirement is that the MOTS is spacetime stable and that energy-momentum contents of the spacetime satisfies the dominant energy condition. It is therefore a very general and robust inequality which reveals a deep connection between the rotation and the shape of quasi-local black holes in four spacetime dimensions.

Theorem 1. *Let (S,h) be an axially symmetric, two-dimensional MOTS, stable with respect to an achronal direction v in a spacetime satisfying DEC. Then*

$$|S| \geq 8\pi |J|. \tag{7}$$

Moreover, equality can only happen if the following five conditions are simultaneously satisfied:

(i) S is marginally stable,
(ii) $h = |J|\left(1+\cos^2\theta\right)d\theta^2 + \frac{4|J|\sin^2\theta}{1+\cos^2\theta}d\varphi^2$,
(iii) $G(\ell,k) = 0$ *on S,*
(iv) $z = \frac{2J\sin^2\theta}{|J|\left(1+\cos^2\theta\right)^2}d\varphi$,
(v) If v is spacelike then $G(\ell,\ell) = 0$ and $\chi_\ell = 0$.

The equality case corresponds to the geometry of the extreme Kerr horizon, i.e. the induced metric on any spacelike section of the degenerate horizon of the Kerr metric satisfying $M^2_{\text{ADM}} = |J_{\text{ADM}}|$. Previous to Theorem 1, the same inequality had been proved by Dain and Reiris [22] for minimal surfaces embedded in maximal slices of a vacuum spacetime

In the following I will describe the basic steps involved in the proof of this result. The argument has two parts. The first one consists in finding an inequality valid for arbitrary functions on S and applying it to an appropriate function defined in terms of the geometry of S. The second step consists in showing that the resulting inequality can be related to the angular momentum, to conclude finally that (7) holds.

Step 1 was accomplished in [23] as a consequence of the spacetime stability of S by performing suitable direct estimates. However, as mentioned before, stable MOTS satisfy general inequalities given by the Rayleigh-Ritz type characterization described in Proposition 1. This allows us to describe step one as follows.

First, using $u = 1$ in the general inequality (3) (notice that for $u = 1$, the solution to equation (4) is $\omega_u = \text{const}$) and the Gauss-Bonnet theorem shows that the genus of the surface must be positive, or else $z_A = 0$, which would imply $J = 0$ and hence a trivial area-angular momentum inequality. So, only the spherically symmetric case needs to be considered. In this setting axisymmetry implies that the divergence-free one form z_A in the Helmholz decomposition of the connection one-form s_A is proportional to the Killing vector η (with its indexes lowered with the metric h). Consequently, the source term in equation (4) vanishes for any choice of axially symmetric u. The corresponding solution is again $\omega_u = \text{const}$. Since restricting the class of functions u in (3) to being axially symmetric cannot decrease the infimum, it follows from stability that

$$\int_S \left(|Du|^2 + \frac{R(h)}{2} u^2 \right) \eta_h \geq \int_S \left(G(\ell, v^\star) + V|\chi_\ell|^2 + |z|^2 \right) u^2 \eta_h$$
$$\geq \int_S |z|^2 u^2 \eta_h, \tag{8}$$

where the second inequality follows because $\left(G(\ell, v^\star) + V|\chi_\ell|^2 \right) u^2$ is non-negative under the assumptions of the theorem. Now, it is simple to see that the metric h can be written in the form (c.f. [22])

$$h = \frac{1}{\cosh^2 \tau} \left(e^{2c} e^{-\sigma(\tau)} d\tau^2 + e^{\sigma(\tau)} d\varphi^2 \right), \quad -\infty < \tau < \infty, \quad 0 < \phi \leq 2\pi, \tag{9}$$

where c is a constant related to the total area by $|S| = 4\pi e^c$. Regularity on the axis of symmetry (i.e. where $\tau \to \pm\infty$) imposes the following asymptotic behaviour on $\sigma(\tau)$,

$$\lim_{\tau \to \pm\infty} \sigma = c, \qquad \lim_{\tau \to \pm\infty} \frac{d\sigma}{d\tau} = 0.$$

Now, a key insight of Dain and Reiris [22] was to use the function $u = e^{-\frac{\sigma}{2}}$ in the analytic inequality (8). This gives an inequality which involves only σ and z_A. The second part of the proof consists in showing that this leads to an inequality involving only J and the area $|S|$ in such a way that all details of the function $\sigma(\tau)$ and $z_A(\tau)$ disappear. To discuss this second part we use a simplified version of an argument due to [21], which, in turn, is a simplification of the original argument in [37] where a the absolute minimum of a renormalized energy of a harmonic map was computed.

First of all we define a function $Y(\tau)$ (up to an arbitrary additive constant) by the equation $z(\eta)\eta_h = \frac{1}{2}\frac{dY}{d\tau} d\tau \wedge d\varphi$. From the definition of angular momentum we have

$$8J = \lim_{\tau \to \infty} \left(Y(\tau) - Y(-\tau) \right).$$

Now, inserting $u = e^{-\frac{\sigma}{2}}$ and the expression for z_A in terms of $Y(\tau)$ in (8) and computing explicitly the curvature scalar for the metric (9) it is straightforward to see that the stability inequality becomes

$$0 \geq \int_S \left[\frac{1}{X^2} \left(\left(\frac{dX}{d\tau} \right)^2 + \left(\frac{dY}{d\tau} \right)^2 \right) - 4 \right] d\tau d\phi \tag{10}$$

where $X(\tau)$ has been defined as $X = |\eta|^2$ (i.e. $X = e^{\sigma} \cosh^{-2}(\tau)$). Now we can view the first term in the integrand as the energy-density of a path $\gamma(\tau) \equiv \{X(\tau), Y(\tau)\}$ in the hyperbolic space $(\mathbb{H}^2, g_{\mathbb{H}^2} = \frac{dX^2+dY^2}{X^2})$. Note that, since the variable τ takes values on the whole real line and $X(\tau) \to 0$ when $|\tau| \to \infty$ the total energy of the path diverges (this simply reflects the fact that while the function in the right-hand side of (10) is integrable on S, this property is obviously not true for the constant term in the expression). Rewriting the total integral as a definite integral from $\tau = -L$ to $\tau = L$ and sending L to infinity, inequality (10) becomes, after performing the trivial angular integration,

$$0 \geq \lim_{L \to +\infty} \left(\int_{-L}^{L} g_{\mathbb{H}^2}(\dot{\gamma}, \dot{\gamma}) \, d\tau - 8L \right),$$

where dot means derivative with respect to τ. We now apply once again the Hölder inequality in the form $\int_{-L}^{L} g_{\mathbb{H}^2}(\dot{\gamma}, \dot{\gamma}) \, d\tau \geq \frac{1}{2L} (\int_{-L}^{L} \sqrt{g_{\mathbb{H}^2}(\dot{\gamma}, \dot{\gamma})} d\tau)^2$ and use the obvious property that the total length of any curve if never smaller than the distance between their initial and final points to rewrite the stability inequality as

$$0 \geq \lim_{L \to +\infty} \left(\frac{1}{2L} \mathrm{dist}^2_{\mathbb{H}^2} [\gamma(L), \gamma(-L)] - 8L \right). \tag{11}$$

Now, the distance function between two arbitrary points (X_1, Y_1) and (X_2, Y_2) in the Poincaré upper-half plane is well-known to be [38]

$$\mathrm{dist}_{\mathbb{H}^2} [(X_1, Y_1), (X_2, Y_2)] = \operatorname{arccosh} \left[1 + \frac{(Y_2 - Y_1)^2 + (X_2 - X_1)^2}{2X_1 X_2} \right].$$

The limit $L \to +\infty$ in (11) is straightforward to obtain after using $\sigma(\pm L) \to c$ and $Y(L) - Y(-L) \to 8J$ and gives

$$0 \geq 8 \ln \left(\frac{8\pi |J|}{|S|} \right) \quad \Longleftrightarrow \quad |S| \geq 8\pi |J|,$$

which proves the inequality. The statements regarding the equality case are also straightforward in this framework and follow by imposing that all inequalities along the process become equalities. In particular, the curve $\gamma(\tau)$ must be a parametrized geodesic in $(\mathbb{H}^2, g_{\mathbb{H}^2})$. This fixes the metric h and z_A to be those in the extreme Kerr geometry. This proves points (ii) and (v) in the theorem. The rest follows directly from the non-negative terms discarded in the stability inequality.

6 Area-angular momentum inequality for black holes

As mentioned in the Introduction, the first examples of area-angular momentum inequalities were obtained for black holes. The assumptions in this setting were that the spacetime is stationary and axially symmetric and that it contains a Killing horizon. Moreover, the spacetime was assumed to be vacuum in a neighbourhood of the horizon, but matter was allowed outside the black hole. The first case treated in the literature assumed the Killing horizon to be degenerate, i.e. with vanishing surface gravity. In this setting, Ansorg and Pfister [15] were able to show that the inequality $|S| = 8\pi|J|$ was universally valid. Here $|S|$ is the area of any axially symmetric spacelike section of the Killing horizon. The non-degenerate case was considered in [16] where the inequality $|S| > 8\pi|J|$ was proved under the additional condition that the black hole is *subextremal*. The definition of subextremal is due to [39] and requires the existence of a future directed null vector k transverse to the Killing horizon for which the inequality $\delta_k \theta_\xi < 0$ holds, where ξ is the Killing vector which generates the Killing horizon. It is clear from the definition of strict stability of MOTS that a Killing horizon is subextremal if and only if all of its spacelike sections are strictly stable along the transverse direction k. The condition of subextremality was used in [16] by working in an Eddington-Finkelstein advanced extension of the following metric in Boyer-Lindquist type coordinates

$$ds^2 = \hat{\mu}\left(\frac{dR^2}{R^2 - r_h^2} + d\theta^2\right) + \hat{u}\sin^2\theta\,(d\varphi - \omega dt)^2 - \frac{4}{\hat{u}}\left(R^2 - r_h^2\right)dt^2,$$

where the Killing horizon is located at $R = r_h$. Imposing the condition $\delta_m \theta_\xi < 0$, where m is proportional to ∂_R in Eddington-Finkelstein coordinates, and discarding a number of positive terms, the actual "subextremality" assumption made by the authors was

$$\int_{S_0} \partial_R(\hat{u}\hat{\mu})|_{R=r_h} \sin\theta d\theta > 0,$$

where S_0 is a section of the Killing horizon corresponding to a constant coordinate time $\hat{t}$ in the Eddington-Finkelstein advanced coordinate system. This equality is certainly implied by the geometric subextremality condition by Booth and Fairhurst, but it is not equivalent to it and, in principle, it is weaker. The purely local inequality in Theorem 1 requires stability, but not strict stability, so the connection between the two inequalities is not immediately obvious. The proofs are also quite different, so it became a problem of interest to try and relate the purely local and the black hole versions of the area-angular momentum inequality. A first step along this direction was made in [21] where the comparison was focused on the relationship between the proof in the minimal surface case in [22] and the behaviour of the Killing horizon on its bifurcation surface. However, no geometrically clear reason of why both inequalities work in seemingly different regimes was given. In [26] a detailed study of the stability properties of Killing horizons was performed. As a by product, a clear relationship between the two types of area-angular momentum inequality was obtained. The next section is devoted to reviewing these results briefly

7 MOTS and Killing horizons

Recall that, in a spacetime (M,g) admitting a Killing vector ξ, a Killing horizon $\mathscr{H}$ is a null hypersurface where the Killing vector ξ is null, tangent and nowhere zero. The integral lines of ξ are null geodesics on $\mathscr{H}$ and the surface gravity is the scalar function on $\mathscr{H}$ defined by κ: $\nabla_\xi \xi \stackrel{\mathscr{H}}{=} \kappa \xi$. It is well-known that κ is constant if (M,g) satisfies the DEC. In the following we will assume that $\mathscr{H}$ has topology $S \times \mathbb{R}$ with S closed and that the $\mathbb{R}$ factor is tangent to the integral lines of ξ.

The Killing equations imply immediately that all spacelike sections S_0 in $\mathscr{H}$ are MOTS and, in fact, with vanishing second fundamental form along ξ, i.e. $\chi_\xi(S_0) = 0$. The Raychaudhuri equation then implies $G(\xi,\xi) \stackrel{\mathscr{H}}{=} 0$. These two properties say that the function W introduced before vanishes identically in this case and, hence, that the stability operator of S_0 is independent of the transverse direction v. Both the stability operator and its principal eigenvalue are therefore properties of S_0 alone. We will denote them simply by L_{S_0} and λ_{S_0} respectively. A natural question is then whether the stability is a property of the horizon itself or whether it depends on the choice of section S_0. To address this issue it is convenient to obtain first an explicit form of the stability operator. This was done in [26] for general totally geodesic null hypersurfaces. Here we restrict ourselves to Killing horizons for definiteness.

Theorem 2. *Let S_0 be a spacelike section of a Killing horizon $\mathscr{H}$ of topology $S \times \mathbb{R}$. Let k be a null vector field on S_0, orthogonal to S_0 and satisfying $\langle \xi, k \rangle = -1$. The stability operator L_{S_0} takes the form*

$$L_{S_0}(\psi) = -D_A \left(u D^A \left(\frac{\psi}{u} \right) \right) + 2 z^A D_A \psi - \kappa \theta_k \psi,$$

where the positive scalar function u and the one-form z_A are defined via the Helmholz decomposition of the normal connection one-form s_A as $s = z + \frac{du}{2u}$.

Notice that u is defined up to an arbitrary multiplicative constant, which obviously has no effect in the expression for L_{S_0}. Combining this theorem with the behaviour of s_A and θ_k under a change of section the following theorem follows [26]

Theorem 3. *If $\mathscr{H}$ has constant surface gravity, then λ_{S_0} is independent of S_0. Moreover there exist Killing horizons $\mathscr{H}$ with non-constant κ for which λ_{S_0} depends on S_0.*

Regarding the analysis of the area-angular momentum inequality for Killing horizons, it turns out that the stability of S_0 has implications on certain integral of the transverse null expansion θ_k on S_0. More precisely [26]

Proposition 2. *Let $\mathscr{H}$ be a Killing horizon with topology $S \times \mathbb{R}$ and let S_0 any spacelike section of $\mathscr{H}$. Assume that z_A and du in the decomposition $s = z + \frac{du}{2u}$ are orthogonal (this occurs automatically if $\mathscr{H}$ is axially symmetric and S_0 respects the axial symmetry and has spherical topology). Then the following holds*

- *If S_0 is stable then $\int_{S_0} \kappa\theta_k u \leq 0$*
- *If S_0 is strictly stable then $\int_{S_0} \kappa\theta_k u < 0$.*
- *If S_0 is stable and κ is constant and non-zero, then $\int_{S_0} \kappa\theta_k u = 0 \Longleftrightarrow \theta_k = 0$.*

This Proposition is the key property that allows one to find the link between the proof of the area-angular momentum inequality in the black hole case [16] and the purely local inequality in [23]. This relationship is given in the following result [26]

Theorem 4. *Assume that the spacetime (M,g) satisfies the DEC and admits a Killing vector ξ with a Killing horizon $\mathscr{H}$. Assume that $\mathscr{H} \simeq \mathbb{S}^2 \times \mathbb{R}$ is axially symmetric with $\kappa \neq 0$ constant. Write the metric of spacelike sections of $\mathscr{H}$ in the form*

$$h = e^{2c} e^{-\sigma} d\theta^2 + e^{\sigma} \sin^2\theta d\phi^2. \tag{12}$$

Then there exists a section S_1 for which $s = z - \frac{1}{2} d\sigma$. Moreover, if $\int_{S_1} \kappa\theta_k e^{-\sigma} \leq 0$ then $|S| \geq 8\pi|J|$.

Notice that S_1 is *not* assumed to be stable in this theorem. The inequality involving θ_k is sufficient for proving the area-angular momentum inequality. Proposition 2 shows that stability of the Killing horizon (or equivalently stability of any of its spacelike sections) is sufficient to imply the validity of the integral inequality assumed in the theorem. It turns out that the surface S_0 used by [16] is *precisely* the surface S_1 in this theorem. Moreover, their inequality $\int_{S_0} \partial_R(\hat{u}\hat{\mu})|_{R=r_h} \sin\theta d\theta > 0$ is precisely the geometric inequality $\int_S \kappa\theta_k e^{-\sigma} < 0$, and the link between the two approaches becomes clear. A similar clarification for the degenerate case has been recently obtained in [40].

8 Further results and open problems

As mentioned in the Introduction, when the spacetime has an electromagnetic field, the area-energy momentum inequality can be strengthened to include the electric charge. This was first done for stationary and axially symmetric degenerate black holes in [15]. The main assumption was that the spacetime is electrovacuum in a neighbourhood of the horizon and it was shown that the equality $|S| = 4\pi\sqrt{Q^4 + 4J^2}$ always holds. In the same setting, but allowing non-degenerate subextremal black holes, the strict inequality $|S| > 4\pi\sqrt{Q^4 + 4J^2}$ was proved in [17]. The local version of this inequality (i.e. for stable, axially symmetric MOTS) has been proved in [25]

Everything we have said so far involves four-dimensional spacetimes. Recent work by Hollands [41] establishes the following generalization to arbitrary dimension.

Theorem 5. *Let (M^{n+1}, g) be a vacuum spacetime with a non-negative cosmological constant Λ, $n \geq 3$ and which admits a stable Killing horizon $\mathscr{H}$ of topology $S \times \mathbb{R}$ with S closed. Assume further that (M^{n+1}, g) admits an additional isometry group $U(1)^{n-2}$ leaving $\mathscr{H}$ invariant and consider a section S_0 of $\mathscr{H}$ respecting the $U(1)^{n-2}$ symmetries. Let $\eta_\pm$ be the Killing vectors which vanish, respectively, at the "north" and "south" poles of S_0. Then*

$$|S_0| \geq 8\pi |J(\eta_+)J(\eta_-)|^{1/2}$$

where $J(\eta) := \frac{1}{8\pi}\int_{S_0} s(\eta)\eta_{S_0}$

(for the precise definition of the vectors $\eta_\pm$ see in [41]).

Before concluding this lecture, let me present a brief list of open problems. The first one refers to the higher dimensional case. The statement above requires that the spacetime contains a Killing horizon and that the spacetime is vacuum, possibly with a positive cosmological constant. It would be of interest to relax this and admit a general spacetime satisfying the DEC. It is also of interest to prove the statement directly at the local level, i.e. for stable MOTS. Another interesting problem in this context is whether the symmetry assumptions can be relaxed to a small number of linearly independent Killing vectors.

In relation to the structure of the proof of the area-angular momentum inequality, it would be of interest to find a deeper reason of why the choice of $u = e^{-\frac{\sigma}{2}}$, where σ is defined as a metric coefficient, is the appropriate choice and what is the underlying reason for the role played by the hyperbolic space (and other highly symmetric spaces in the charged case) in the proof of the inequality.

Another interesting problem is to understand why the surface $\tilde{t} = \text{const}$ in Eddington-Finkesltein coordinates for black holes is precisely the surface S_1 in the area-angular momentum inequality for Killing horizons, i.e. why for these coordinates, the normal connection one-form is linked to the induced metric via the condition spelled out in Theorem 4.

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