Backreaction Effects on the Luminosity-Redshift Relation in Inhomogeneous Cosmology

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Abstract We recall a general gauge invariant formalism for defining cosmological averages that are relevant for observations based on light-like signals. Using such formalism, together with adapted "geodesic light-cone" coordinates, the effect of a stochastic background of cosmological perturbations on the luminosity-redshift relation is computed to second order. The resulting expressions are free from both ultraviolet and infrared divergences, implying that such perturbations cannot mimic a sizable fraction of dark energy. Different averages are estimated and depend on the particular function of the luminosity distance being averaged. The energy flux, being minimally affected by perturbations at large z, is proposed as the best choice for precision estimates of dark-energy parameters. Nonetheless, its irreducible (stochastic) variance induces statistical errors on $\Omega_{\Lambda}(z)$ typically lying in the few-percent range.

1 Introduction

Establishing the existence of dark energy and determining its parameters is one of the central issues in modern cosmology. Evidence for a sizable dark-energy component in the cosmic fluid comes from different sources: CMB anisotropies, models of large-scale-structure formation and, most directly, the luminosity redshift relation of Type Ia supernovae, used as standard candles.

In this latter case, on which we concentrate our attention, the analysis is usually made in the simplified context of a homogeneous and isotropic (FLRW) cosmology. The issue has then been raised about whether inhomogeneities may affect the

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conclusion of such a naive analysis. In particular, one should address this question in the presence of stochastically isotropic and homogeneous perturbations of the kind predicted by inflation. In such a context, the possibility that sub-horizon perturbations may simulate a substantial fraction of dark energy, or that they may at least play some role in the context of near-future precision cosmology, has to be seriously considered.

In order to address these issues we follow [1] and study the luminosity-redshift relation in a spatially-flat ΛCDM model. The luminosity distance d_L now depends on the redshift z as well as on the angular coordinates of the sources, and must be inserted in an appropriate light-cone and *ensemble* average [2, 3].

The paper is organized as follows. In Section 2 we briefly present the prescription used to average on null hypersurfaces. In Section 3 we discuss the effect of a stochastic background of inhomogeneities on different functions of the luminosity-redshift relation. Our conclusive remarks are then presented in Section 4.

2 Gauge invariant light-cone averaging

Let us here briefly present a general gauge invariant formalism for defining cosmological averages that are relevant for observations based on light-like signals. Following [2], we start with a spacetime integral where the four-dimensional integration region is bounded by two hypersurfaces, one spacelike and the other one null (corresponding e.g. to the past light-cone of some observer). Let us choose, in particular, the region inside the past light-cone of an observer bounded in the past by the hypersurface defined by $A(x) = A_0$: clearly a gauge invariant definition of the integral of a scalar $S(x)$ over such a hypervolume can be written as

$$I(S;-;A_0,V_0) = \int_{\mathscr{M}_4} d^4x\sqrt{-g}\;\Theta(V_0-V)\Theta(A-A_0)S(x), \qquad (1)$$

where $V(x)$ is a scalar satisfying $\partial_\mu V \partial^\mu V = 0$, and where the "$-$" symbol on the l.h.s. denotes the absence of delta-like window functions.

Starting with this hypervolume integral we can construct covariant and gauge invariant hypersurface and surface integrals considering the variation of the volume average along the flow lines n_μ normal to the reference hypersurface $\Sigma(A)$ defined by $A(x)$ equal to a constant.

Considering the variation of the hypervolume integral by shifting the light-cone $V = V_0$ along n_μ, we obtain the integral on the past light-cone itself starting from a given hypersurface in the past

$$I(S;V_0;A_0) = \int d^4x\sqrt{-g}\,\delta(V_0-V)\Theta(A-A_0)\frac{|\partial_\mu V \partial^\mu A|}{\sqrt{-\partial_\nu A \partial^\nu A}}S(x)\,. \qquad (2)$$

While considering the variation of the hypervolume integral both by shifting the light-cone $V = V_0$ and the hypersurface $A = A_0$ along n_μ we obtain the integral on the 2-sphere embedded in the past light-cone

$$I(S;V_0,A_0;-) = \int d^4x\sqrt{-g}\,\delta(V_0-V)\delta(A-A_0)|\partial_\mu V\partial^\mu A|S(x)\,. \tag{3}$$

We note, finally, that averages of a scalar S over different (hyper)surfaces are trivially defined by:

$$\langle S\rangle_{V_0,A_0} = \frac{I(S;V_0,A_0;-)}{I(1;V_0,A_0;-)}\;; \tag{4}$$

$$\langle S\rangle^{A_0}_{V_0} = \frac{I(S;V_0;A_0)}{I(1;V_0;A_0)}\;; \tag{5}$$

3 Backreaction on the luminosity-redshift relation

Let us start by recalling the standard expression for the luminosity distance in an unperturbed flat ΛCDM model, with present fractions of critical density Ω_Λ and $\Omega_m = 1-\Omega_\Lambda$:

$$d_L^{FLRW}(z) = \frac{1+z}{H_0}\int_0^z \frac{dz'}{[\Omega_\Lambda+\Omega_m(1+z')^3]^{1/2}}. \tag{6}$$

Consider now the expression for d_L in the corresponding perturbed geometry. Combining light-cone and ensemble averages (denoted, respectively, by brackets and over-bars), we can write the averaged result in the form:

$$\overline{\langle d_L\rangle}(z) = d_L^{FLRW}\,[1+f_d(z)]\,, \tag{7}$$

where $f_d(z)$ represents the "backreaction" on d_L due to inhomogeneities. For consistency, d_L has to be computed (at least) up to the second perturbative order since *ensemble* averages of first-order quantities are vanishing for stochastic perturbations. A detailed computation of $f_d(z)$ would thus enable to extract the "true" value of the dark-energy parameters from the measurement of $\overline{\langle d_L\rangle}(z)$ after taking the correction into account.

However, as already stressed in [3], given the covariant (light-cone) average of a perturbed (inhomogeneous) observable S the average of a generic function of this observable differs, in general, from the function of its average, i.e. $\overline{\langle F(S)\rangle} \neq F(\overline{\langle S\rangle})$. Expanding the observable to second order as $S = S_0+S_1+S_2+\cdots$, one finds:

$$\overline{\langle F(S)\rangle} = F(S_0)+F'(S_0)\overline{\langle S_1+S_2\rangle}+F''(S_0)\overline{\langle S_1^2/2\rangle} \tag{8}$$

where $\overline{\langle S_1\rangle} \neq 0$ as a consequence of the "induced backreaction" terms (see [3]). Thus different functions of the luminosity distance are differently affected by the inhomogeneities, and require different "subtraction" procedures. Finding the function that minimizes the backreaction will help of course for a precision estimate of the cosmological parameters.

The average value of Φ, obviously controlled by the average of d_L^{-2}, has to be carried out on the past light-cone of the observer, at a fixed redshift z, using the gauge-invariant prescription described. This is most conveniently done [2, 3] in the so-called geodesic light-cone gauge (GLC), where the metric depends on six arbitrary functions ($\Upsilon, U^a, \gamma_{ab}$, $a,b=1,2$), and the line-element takes the form (with $\widetilde{\theta^1}=\widetilde{\theta}, \widetilde{\theta^2}=\widetilde{\phi}$):

$$ds^2=\Upsilon^2 dw^2-2\Upsilon dw d\tau+\gamma_{ab}(d\widetilde{\theta}^a-U^a dw)(d\widetilde{\theta}^b-U^b dw). \tag{9}$$

In the GLC gauge the past light-cone is defined by the condition $w=w_0=$ const, and the redshift is given by:

$$1+z=\Upsilon(w_0,\tau_0,\widetilde{\theta}^a)/\Upsilon(w_0,\tau,\widetilde{\theta}^a). \tag{10}$$

Furthermore, the luminosity distance of the source is simply expressed as [3] $d_L=(1+z)^2\gamma^{1/4}(\sin\widetilde{\theta})^{-1/2}$, yielding the following exact result [4]:

$$\langle d_L^{-2}\rangle(z,w_0)=\frac{4\pi(1+z)^{-4}}{\int d^2\widetilde{\theta}^a\sqrt{\gamma(w_0,\tau(z,\widetilde{\theta^a}),\widetilde{\theta}^b)}}, \tag{11}$$

where $\gamma=\det\gamma_{ab}$, and $\tau(z,\tilde{\theta}^a)$ is obtained by solving Eq. (10). The above expression has a simple physical interpretation: the averaged flux, for a given z, is inversely proportional to the proper area of the surface lying on our past light-cone at the given value of z.

To compute this quantity in the perturbed geometry of our interest, we need to express it in a gauge where the stochastic background of cosmological perturbations is explicitly known up to second order. To this purpose, we can use the standard Poisson gauge where we include first and second-order scalar perturbations, neglecting their tensor and vector counterparts. Performing the relevant transformations to second order we arrive at the following analogue of (7):

$$\overline{\langle d_L^{-2}\rangle}=(d_L^{FLRW})^{-2}\overline{(I_\Phi(z))^{-1}}\equiv(d_L^{FLRW})^{-2}\left[1+f_\Phi(z)\right], \tag{12}$$

where I_Φ has in general the following structure:

$$I_\Phi(z)=\int\frac{d\tilde{\phi}d\tilde{\theta}\sin\tilde{\theta}}{4\pi}\Big[1+\mathscr{I}_1+\mathscr{I}_{1,1}+\mathscr{I}_2\Big](\tilde{\theta},\tilde{\phi},z). \tag{13}$$

Here $\mathscr{I}_1$, $\mathscr{I}_{1,1}$, $\mathscr{I}_2$ are, respectively, the first-order, quadratic first-order, and genuine second-order contributions of our stochastic fluctuations. After solving the relevant perturbation equations [5] they can all be expressed in terms of the first-order Bardeen potential $\Psi(x,\eta)$. Using the stochastic properties of this perturbation, and expanding in Fourier modes $\Psi_k(\eta)$, we can then obtain an expression for

$\overline{(I_\Phi)^{-1}}$ where the scalar perturbations only appear through the so-called dimensionless power spectrum, $\mathscr{P}(k,\eta) = (k^3/2\pi^2)|\Psi_k(\eta)|^2$.

Considering a ΛCDM model we have to proceed with an approximate numerical integration. The result can be then written in the form

$$f_\Phi(z) = \int_0^\infty \frac{dk}{k}\,\mathscr{P}(k)\Big[f_{1,1}(k,z) + f_2(k,z)\Big]. \tag{14}$$

At leading order the contribution, in the region of z relevant for dark-energy phenomenology, comes from terms of the type $f(k,z) \sim (k/\mathscr{H}_0)^2 \widetilde{f}(z)$, and we can write, to a very good accuracy,

$$f_\Phi(z) \simeq \Big[\widetilde{f}_{1,1}(z) + \widetilde{f}_2(z)\Big] \int_0^\infty \frac{dk}{k}\left(\frac{k}{\mathscr{H}_0}\right)^2 \mathscr{P}(k). \tag{15}$$

The absolute value (and sign) of $f_\Phi(z)$ is illustrated in Fig. 1, showing that the backreaction of a realistic spectrum of stochastic perturbations induces negligible corrections to the averaged flux at large z. In addition, such corrections have the wrong z-dependence (in particular change sign at some z) to simulate even a tiny dark-energy component. For the considered spectrum (behaving as $k^{n_s-5}\log^2 k$ at large k, see [6]) the spectral integral is convergent and very weakly sensitive to the chosen value of the UV cutoff [3] representing here the limit of validity of our perturbative approach.

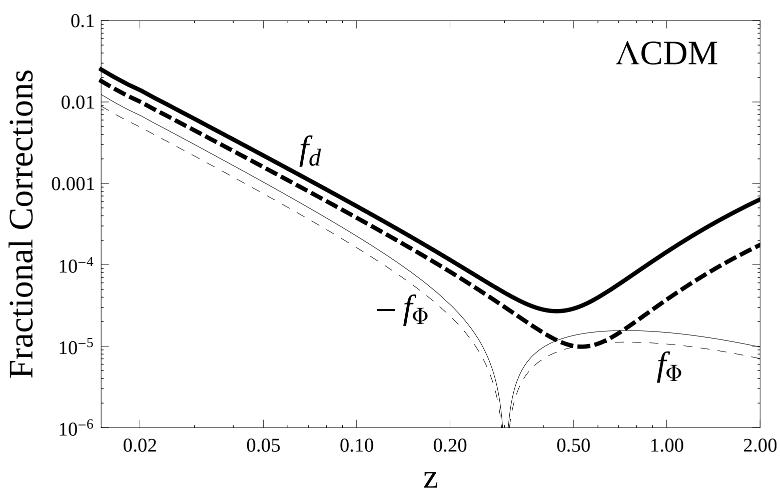

Fig. 1 The correction f_Φ of Eq. (12) (thin curves) is compared with the correction f_d of Eq. (16) (thick curves), for a ΛCDM model with $\Omega_\Lambda = 0.73$. We have used two different cutoff values: $k_{UV} = 0.1\text{Mpc}^{-1}$ (dashed curves) and $k_{UV} = 1\text{Mpc}^{-1}$ (solid curves). We have used for $\mathscr{P}(k)$ the inflationary scalar spectrum with the WMAP parameters [7] and the transfer function given in [6] (see also [3]).

The small value of $|f_\Phi|$ at large z leads us to conclude that the averaged flux is a particularly appropriate quantity for extracting from the observational data the "true" cosmological parameters. On the other hand, the situation is somewhat different for other functions of d_L.

Indeed, let's apply the general result (8) to the flux variable, $S = \Phi$, and consider two important examples: $F(\Phi) = \Phi^{-1/2} \sim d_L$, and $F(\Phi) = -2.5\log_{10}\Phi + \text{const} \sim \mu$ (the distance modulus). For d_L, following the notations of Eq. (7) and using the general result (8), we obtain:

$$f_d = -(1/2)f_\Phi + (3/8)\overline{\langle(\Phi_1/\Phi_0)^2\rangle}. \tag{16}$$

Similarly, for the distance modulus we obtain:

$$\overline{\langle\mu\rangle} - \mu^{FLRW} = -1.25(\log_{10} e)\left[2f_\Phi - \overline{\langle(\Phi_1/\Phi_0)^2\rangle}\right]. \tag{17}$$

As clearly shown by the two above equations, the corrections to the averaged values of d_L and μ are qualitatively different from those of the flux, because of the extra contribution (inevitable for any non-linear function of the flux) proportional to the square of the first-order fluctuations. The averaged flux corrections have leading spectral contributions of the type $k^2\mathscr{P}(k)$; on the contrary, the new corrections to d_L and μ are due to the so-called "lensing effect", they dominate at large z, and have leading spectral contributions of the type $k^3\mathscr{P}(k)$ (as already discussed in [3]). The explicit numerical integration, reported in Fig. 1, confirms that $|f_\Phi| \ll f_d$ at large z. We stress that even the k^3-enhanced contributions are UV-finite for the case under consideration.

Let us now briefly discuss to what extent the enhanced corrections due to the squared first-order fluctuations can affect the determination of the dark-energy parameters if quantities other than the flux are used in the fits. To this purpose we consider the much used (average of the) distance modulus given in Eq. (17), referred as usual to the homogeneous Milne model with $\mu^M = 5\log_{10}[(2+z)z/(2H_0)]$. In Fig. 2 we compare the averaged value $\overline{\langle\mu\rangle} - \mu^M$ with the corresponding expression in a homogeneous ΛCDM model with different values of Ω_Λ. We also show the expected dispersion around the averaged result, represented by the square root of the variance [3]. The latter is given by:

$$\sqrt{\overline{\langle\mu^2\rangle} - \left(\overline{\langle\mu\rangle}\right)^2} = \pm 2.5(\log_{10} e)\sqrt{\overline{\langle(\Phi_1/\Phi_0)^2\rangle}}; \tag{18}$$

while for the flux we simply find:

$$\sqrt{\overline{\langle(\Phi/\Phi_0)^2\rangle} - \left(\overline{\langle\Phi/\Phi_0\rangle}\right)^2} = \pm\sqrt{\overline{\langle(\Phi_1/\Phi_0)^2\rangle}}. \tag{19}$$

As illustrated in Fig. 2, we find that, even for the distance modulus, the effect of inhomogeneities on the average only affects the determination of Ω_Λ at the third decimal figure (see also Fig. 1), at least for the inflationary power spectrum with the ΛCDM transfer function of [6]: in that case, the curves for $\overline{\langle\mu\rangle}$ and $\mu^{\rm FLRW}$ are practically coincident at large z.

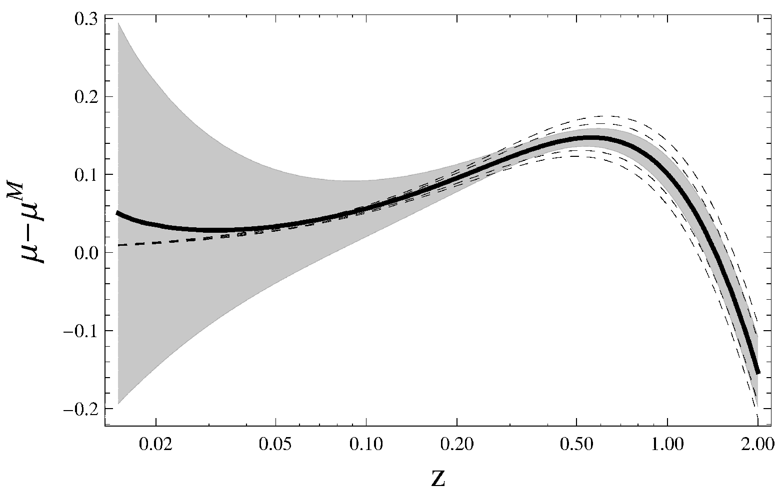

Fig. 2 The averaged distance modulus $\overline{\langle\mu\rangle} - \mu^M$ (thick solid curve), and its dispersion of Eq. (18) (shaded region) are computed for $\Omega_\Lambda = 0.73$ and compared with the homogeneous value for the unperturbed ΛCDM models with $\Omega_\Lambda = 0.69, 0.71, 0.73, 0.75, 0.77$ (dashed curves). We have used $k_{UV} = 1{\rm Mpc}^{-1}$ and the same spectrum as in Fig. 1.

4 Conclusions

The main results presented in this paper can be summarized as follows. We have recalled a covariant and gauge invariant formalism to average on null hypersurfaces and to analyze the effects of inhomogeneities on astrophysical observables related to light-like (massless) signals.

Then we have seen how the gauge invariant light-cone averaging of the luminosity-redshift relation leads to results which are automatically free from UV/IR divergences for *any* function of the luminosity distance, and, as a consequence, cannot simulate a substantial fraction of dark energy.

The actual value of the backreaction strongly depends on the quantity being averaged. It turns out to be minimal for the flux which, therefore, stands out as the safest observable for precision cosmology. For other observables the backreaction is instead considerably larger.

The dispersion due to stochastic fluctuations is much larger than the backreaction itself, implying an irreducible scatter of the data that may limit to the percent level (see Fig. 2) the precision attainable on cosmological parameters because of the present limited statistics.

References

1. I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier, G. Veneziano, *Do stochastic inhomogeneities affect dark-energy precision measurements?*, Physical Review Letters **110**(2), 021301 (2013)
2. M. Gasperini, G. Marozzi, F. Nugier, G. Veneziano, *Light-cone averaging in cosmology: Formalism and applications*, J. Cosmol. Astropart. Phys. **2011**(07), 008 (2011)
3. I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier, G. Veneziano, *Backreaction on the luminosity-redshift relation from gauge invariant light-cone averaging*, J. Cosmol. Astropart. Phys. **2012**(04), 036 (2012)
4. I. Ben-Dayan, G. Marozzi, F. Nugier, G. Veneziano, *The second-order luminosity-redshift relation in a generic inhomogeneous cosmology*, J. Cosmol. Astropart. Phys. **2012**(11), 045 (2012)
5. N. Bartolo, S. Matarrese, A. Riotto, *The full second-order radiation transfer function for large-scale CMB anisotropies*, J. Cosmol. Astropart. Phys. **2006**(05), 010 (2006)
6. D.J. Eisenstein, W. Hu, *Baryonic features in the matter transfer function*, Astrophys. J. **496**, 605 (1998)
7. E. Komatsu, K.M. Smith, J. Dunkley, et al., *Seven-year Wilkinson microwave anisotropy probe (WMAP) observations: Cosmological interpretation*, Astrophys. J. Suppl. Ser. **192**, 18 (2011)