Granularity in Angle: Observability in Scattering Experiments

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Abstract Geometry is quantized in loop quantum gravity. As a step toward building a detailed phenomenology of this discrete geometry a model of an atom of geometry is reviewed. The model, which preserves local Lorentz invariance, exhibits a lever arm that raises the scale at which the granularity in angle becomes apparent. The signature of this effect is a systematic shift of observed angles in processes such as high energy particle scattering experiments. To check assumptions in the model, coherent states of a simple atom of spatial geometry are explored using information intrinsic to the quantum state.

1 Introduction

If space-time or spatial geometry is fundamentally discrete, it will be observationally manifest. The kinematics of Loop Quantum Gravity (LQG) predicts discrete spectra of spatial geometric quantities such as volume, angle, and length. Before the dynamics, and the quantization, is complete we do not know whether the kinematic results extend to the physical state space [1, 2]. However, in the absence of a complete theory and even *because* the complete theory is not yet finished, it is useful to know how the predicted discreteness in spatial geometry could be manifest in observation.

Perhaps the first reaction to granular geometry is that the theory breaks Lorentz invariance. Certainly broken Lorentz symmetry leads to dramatic effects that produce strong constraints (see [3] and [4]), but broken local Lorentz invariance does not necessarily follow from discreteness – the discrete nature of quantum angular

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momentum does not imply the loss of rotational invariance. Furthermore, the relative velocity relating inertial reference frames depends on the metric and therefore the measurements of areas of a surface in different frames are not directly comparable [5]. More directly the causal set approach to quantum gravity shows that Lorentz invariance and (space-time) discreteness are compatible.

Without the blunt effects of breaking local Lorentz invariance are there other effects that might reveal the fundamental discreteness of spatial geometry? This contribution reviews a model based on the kinematic predictions of discrete spatial geometry in LQG and demonstrates that the answer is in the affirmative. The model does not break local Lorentz invariance and yet there are effects at a mesoscopic scale above the Planck scale [6]. The 'lever arm' that raises the scale is due to the underlying asymmetry of the angle operator spectrum and the combinatorics of an atom of spatial geometry. At this mesoscopic scale the local geometry differs from flat three dimensional space leading to systematic shift in the distribution of measured angles. To make such a model at this stage of development of LQG and QFT in such a context requires assumptions. These are detailed in section 3.

To begin to test these assumptions we explore coherent states based on the 'hydrogen atom of spatial geometry', a single 4-valent node. Within the limitations of this model we still find a hint of a lever arm, although in this context it is purely due to the uncertainty relations among non-commuting operators.

2 Angle Operator

The angle operator was originally defined in [7] and described further in[6, 8, 9]. In this contribution we focus on the spatial atom. So, briefly, the angle operator is defined on a truncation of the full kinematic Hilbert space, at a single spin network node. The incident links to this node are partitioned into three sets, C_1, C_2, and C_3. Three left-invariant gravitational field operators L^i_1, L^i_2, and L^i_3 are defined by these partitions. (The notation is follows [10].) In terms of these gravitational field operators for the partitions C_1 and C_2, the quantum angle operator is

$$\hat{\theta}_{(12)} := \arccos \frac{L^i_1 L^i_2}{|L_1|\,|L_2|}, \tag{1}$$

in which $|L| = \sqrt{L^2}$. The sum $\sum_{k=1}^{3} L^i_k$ vanishes due to gauge invariance and the exhaustive partitions. The partitioning of links incident to the node selects classes of preferred intertwiner bases. These are labeled by trivalent nodes where each leg of the node, or branch, connects all links in a single partition. This node is the "*intertwiner core*". On this class of bases the spectrum of the angle operator is

$$\hat{\theta}_{(12)} \,|\, j_1\, j_2\, j_3\rangle = \theta_{(12)} \,|\, j_1\, j_2\, j_3\rangle \text{ with} \tag{2}$$

$$\theta_{(12)} = \arccos\left(\frac{j_3(j_3+1) - j_1(j_1+1) - j_2(j_2+1)}{2\,[j_1(j_1+1)\, j_2(j_2+1)]^{1/2}}\right). \tag{3}$$

For more detail see [6, 9]. With this angular spectrum there are two aspects of flat spatial geometry that are hard to model. The spectrum is sparse for small angles and the frequencies of eigenvalues is far from the $\sin\theta$ distribution of polar angles in flat, 3-dimensional continuum geometry. These are both manifestations of the asymmetry in the angular spectrum. This asymmetry persists even when the spins are very large [11, 8].

It is convenient to visualize the action of the angle operator on polyhedra, with faces dual to the incident links [12]. The areas on the dual surfaces are simply related to the spin j via $\ell_{Pl}\sqrt{j(j+1)}$. The partitioning of C_1, C_2, and C_3 induces a partition of the dual surface into three surface areas S_1, S_2, and S_3. The intertwiner core then represents a decomposition of the polyhedron with "internal faces" determined by j_i.

The notation is as follows. Twice the sum of the representations on the links incident to the node in partition C_k is denoted by the "flux" s_k also denoted $\mathbf{s}$. In the dual surface picture this is the flux of spin through the respective surfaces, roughly equal to the face areas. The total flux is the sum of the spins on all the incident edges, denoted s. The quantities $n_k = 2j_k$, the internal areas, uniquely specify the intertwiner core, denoted $|\mathbf{n}\rangle$. The fluxes s_k and core labels n_k are distinct and satisfy $n_k \leq s_k$.

3 Combinatorial phenomenology

The phenomenological model of an atom of 3-geometry is based on the state space described above and three, additional assumptions [6]: (1) The probability measure on the space of intertwiner cores is uniform. (2) All incident links to the node are spin-$1/2$. (3) The fluxes are large and semi-classical, $1 \ll s_k \ll s_3$, $k = 1,2$. The last assumption is motivated by the numerical studies of [11, 8] showing that the asymmetry in the angular spectrum shifts the distribution away from the usual $\sin\theta$ distribution of polar angles. To recover the classical distribution it was necessary in these studies to take large fluxes, and, in particular $1 \ll s_j \ll s_3$, $j = 1,2$. Fluxes $\mathbf{s}$ that satisfy these relations are called "semi-classical fluxes". This means that we omit terms $O(1/s_i)$, $O(1/n_i)$, and $O(n_i/s_i)$. The assumption (2) is for simplicity, although in statistical studies of large closed surfaces built from oriented areas the spins are about one on average [13].

All physical processes involving angle that we currently observe are on very large scales, many orders of magnitude above the Planck scale. These processes occur in a large effective volume. Since volume scales as the (total flux)$^{3/2} \equiv s^{3/2}$, the scaling

defines an effective length $\ell_s = \ell_{Pl} s^{3/2}$ or energy $M_s = M_{Pl}/\sqrt{s}$ scale. The scale is set in the reference frame used in the experimental analysis, such as the CM frame.

With these assumptions, the combinatorics of the model of a spatial atom can be solved analytically [6]. The combinatorics for the number of states may be simply related to a path counting problem with a known solution [11]. For single branch of the basis $|\mathbf{n}\rangle$, the probability of an internal spin (or face area) n given a total flux of s is

$$p_s(n) = \frac{n}{s} \exp\left(-\frac{n^2}{2s}\right). \tag{4}$$

This is the Rayleigh distribution for a "distance" n covered in $2s$ steps in an isotropic random walk with unit step size in *two* spatial dimensions. The total probability for the internal state of the atom, $p_{\mathbf{s}}(\mathbf{n})$, is just the product of three of these combinatorial factors. For large, semi-classical spins the normalized probability distribution is simply expressed as

$$p_{\mathbf{s}}(\theta) = \sum_{\mathbf{n}} \delta(\theta - \theta(n)) p_{\mathbf{s}}(\mathbf{n}). \tag{5}$$

The partition fluxes $\mathbf{s}$ determine a mixed state, $\rho_{\mathbf{s}} = \sum_{\mathbf{n}} p_{\mathbf{s}}(\mathbf{n}) P_{|\mathbf{n}\rangle}$, where $P_{|\mathbf{n}\rangle}$ is the projector on the orthonormal basis of the intertwiner core. The sum is over the admissible integers $\mathbf{n}$ such that $n_i \leq s_i$. The projector is $P_{|\mathbf{n}\rangle} = |\,\theta_I\rangle\langle\theta_I\,|$ where $|\,\theta_I\rangle = \sum_{\mathbf{n}} c_{\theta_I}(\mathbf{n})\,|\mathbf{n}\rangle$. The probability of finding the angle eigenvalue θ_I in the mixed state $\rho_{\mathbf{s}}$ is

$$\text{Prob}(\theta = \theta_I; \rho_{\mathbf{s}}) = tr(\rho_{\mathbf{s}} P_{\theta_I}) = \sum_{\mathbf{n}} p_{\mathbf{s}}(\mathbf{n}) |\langle n|\,\theta_I\rangle|^2 \equiv p_{\mathbf{s}}(\theta). \tag{6}$$

This procedure can be used to calculate $p_{\mathbf{s}}(\theta)$ for semi-classical fluxes

$$P_{\mathbf{s}}(\theta) := \int d^3n\, p_{\mathbf{s}}(\mathbf{n}) |c_\theta(\mathbf{n})|^2 \delta(\theta - \theta(n)). \tag{7}$$

The integration of equation (7) is straightforward [6]. The key step in the calculation is the identification of the "shape parameter" $\varepsilon := \sqrt{s_1 s_2}/s_3$ that measures the asymmetry in the distribution of angles. As $\varepsilon \to 0$ the continuum distribution of polar angles is recovered.

The resulting angular measure, when expressed in terms of Legendre polynomials, and to $O(\varepsilon^3)$, is [6]

$$\rho_\varepsilon(\theta) \simeq \sin\theta \left(1 - \frac{8}{\pi} P_1(\cos\theta)\varepsilon + \frac{3}{2} P_2(\cos\theta)\varepsilon^2\right). \tag{8}$$

The affect of the modified distribution of polar angles is that the "shape" of space is altered by the atom; the local angular geometry differs from flat 3-space. The total flux s determines the 3-volume of the spatial atom and thus an effective mesoscopic length scale, $\ell_s = \sqrt{s}\ell_P$, greater than the fundamental discreteness scale of ℓ_P. While the the shape parameter ε is free of the Planck scale, the effective length scale, determined by the total fluxes s, is tied to the discreteness scale of the theory. For

instance, a shape parameter $\varepsilon \sim 10^{-3}$ requires a total flux of at least $s \sim 10^6$, raising the length scale at which this asymmetry would be observed to be nine orders of magnitude above the Planck scale. Any angular measurement involving processes at this scale would be affected by this modified distribution. Thus the angle spectrum and the combinatorics of the intertwiner together provide a lever arm that lifts the fundamental scale of the quantum geometry up to a larger, mesoscopic scale.

4 Example: Scattering

Let's sketch how measurement of angle in an effective atom of geometry for a scattering experiment works. (More detail may be found in [6].) Bhabha scattering is convenient because the e^+e^- scattering process involves "point-like" fundamental particles. The scattering cross section depends on angle and, in a theory that encompasses the quantum state of the geometry, scattering events are measurements of the states that support the geometry. The affects of the discrete geometry will be evident at some energy, modifying the QED vertex. Short distance modifications to QED may be expressed in the Drell parameterization [14], which allows modifications to "switch on" at CM energies corresponding to the short distance structure at the scale ℓ_s. Two kinematic effects were studied in [6], one due to the averaging over angle and the other due to the modified distribution of angle in the state described above. The latter effect is dominant. Assuming that the spatial geometry is homogenous so that each scattering event occurs in the same state described in 3 then the Bhabha scattering cross section is, using the Drell parameterization, [6]

$$\left(\frac{d\sigma}{d\Omega}\right) / \left(\frac{d\sigma}{d\Omega}\right)_{QED} \simeq 1 \mp \left(\frac{3s}{\Lambda_{\pm}^2}\right) \left(\frac{\sin^2\theta}{3+\cos^2\theta}\right) \left(1 + \frac{8}{\pi}\cos(\theta)\varepsilon + \ldots\right). \quad (9)$$

A comparison between the model and the data, discussed in [6], shows that the shape correction reduces the observed differential cross section at small angles and increases it at large angles; the shape effect yields a systematic shift in the data.

To check the robustness of this prediction we must both check the framework and the assumptions. In the next section we report on a check of the assumptions of the model and study the most simple atom of geometry, a 4-valent node. We replace the assumption of uniform probability with minimum relative uncertainty and allow higher spin. This allows us to answer the question, Is there evidence of a lever arm in the simplest 4-valent atom of geometry?

5 Coherent states and angle

To check the robustness of the above model we developed coherent states for the simple 4-valent node, dual to a tetrahedron. Coherent states for semi-classical ge-

ometries are based on (some flavor of) $SU(2)$ coherent states, states that are peaked around given directions, $\hat{n}$, normal to the faces. These coherent states are sharply peaked on the scalar products and thus angles, even for moderately large spin, $j > 100$. With such low spin there is no significant lever arm. However the coherent states require classical information, the directions $\hat{n}$. To avoid adding additional assumptions about these classical directions to our model, we constructed coherent states only from quantum information intrinsic to the atom of geometry.

It has been known for some time that the state of the atom contains enough information to establish dimension of space and the scalar products. The spin geometry theorem of Penrose and Moussouris [15, 16] states that for low relative uncertainties – large spin – the state of the geometric atom yields vectors in Euclidean 3-space. Furthermore, the proof is constructive. By minimizing the relative uncertainties, the distribution of directions $\hat{n}$ is determined. We can construct states that model semi-classical geometry using only information intrinsic to the atom [17].

In this construction the directions in 3-space are well-defined when the relative uncertainties of scalar products are minimized. More precisely, when the state is "δ-classical" $\langle \Delta L_k \cdot L_l \rangle / j_k j_l < \delta$, pairs k, l, then directions associated to the faces may be defined. The angles between normals are determined by the scalar products of these vectors. The spin geometry theorem states that there exists a δ for any approximation of vectors 3-dimensional Euclidean space. We constructed these vectors for a tetrahedron.

We set the maximum spin, typically 20, and took a superposition over all states of the atom of geometry, with amplitudes determined by a complex gaussian. We fixed the parameters of the state (peak value, width, and phase) by minimizing the relative uncertainties. This analysis is straightforward in the basis of one of the angle operators. For the other angles, which do not commute with one another, one must employ recoupling theory. As this results in a lengthy expression to minimize, we used Mathematica to minimize the relative uncertainties.

This process of minimizing δ-constraints for the independent angles yields well-defined directions but low-volume, "squashed" geometries. This is expected since the minimization procedure essentially extremizes the cosine of the angle between outgoing normals times a spin factor, held fixed by our choice of maximum spin. Minimizing the resulting sine yields angles are near 0 and π, producing an elongated shape of the atom ("squashed"), with angles near 0 and π. The minimization produces a distribution of angles that is far from the distribution of classical polar angles.

However simultaneously minimizing the relative uncertainties and maximizing the volume of the atom produces a distribution of angles peaked around the center of the classical distribution, at the cost of increased relative uncertainty in the angle (i.e. larger δ's and a correspondingly worse approximation to the angles of the classical tetrahedron). We see that the "cost" of modeling the classical distribution of angles is an increase in relative uncertainty. As we know from the spin geometry theorem to reduce the uncertainty further we need to increase the total flux, raising the effective scale of the atom of geometry. This "lever arm" is simply a manifes-

tation of the familiar uncertainty relations of non-commuting operators, rather than from the combinatorics of more complex states.

This tetrahedron study shows that (1) It is possible to use quantum information intrinsic to the state to define the coherent states. (Unfortunately the computation grows unwieldily for higher spin and high valence atoms.) (2) There is a "short" lever arm in that the classical distribution of angle is achieved for atoms above the Planck scale. But this lever arm is too short to raise the scale of the effects into experimentally accessible regimes. However, the lever arm in 3 arises from the combinatorics arising in complex atoms atoms of geometry.

6 Discussion

This contribution reviews a model that explores effects arising from combinatorial structures in the deep spatial quantum geometry of LQG [6]. The model, based on assumptions in 3, relies on the combinatorics of a discrete model of spatial geometry, a single atom of spatial geometry, the spin network node. This model shows that potentially observable effects of quantum geometry need not be tied to violations of local Lorentz symmetry and that a scale above the fundamental scale of the theory can arise out of the combinatorics of the state.

To test the assumption of the uniform measure we developed coherent states based on information intrinsic to the simplest atom, which is dual to a tetrahedron. Using the spin geometry theorem we developed coherent states of the polyhedron and found evidence for a lever arm. In this case the short lever arm is due to the non-commutivity of operators, rather than the combinatorics of the state. However, this is a very simple atom of geometry that does not have the combinatoric richness of complex higher valence atoms. Work on modeling these more complex structures is ongoing.

Finally, it is important to note that work remains on modeling the vertex modifications in field theory. One possibility is to model the effective metric in, e.g.

$$\mathscr{L}'_{\delta}(x) = -e \int d^4x \bar{\psi}(x)\gamma_\mu \psi(x) g_{\delta}^{\mu\nu}(x-z)A_\nu(z). \tag{10}$$

using spin foam techniques. It remains to be seen whether this shape-corrected QED vertex yields the simple shape corrections discussed here.

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