Exact Dynamical AdS Black Holes and Wormholes with a Klein-Gordon Field

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Abstract We present an exact solution with spherical, plane, or hyperbolic symmetry in the Einstein-Klein-Gordon system with negative Λ in arbitrary dimensions. In the coordinate system we adopt, the scalar field is homogeneous and the spacetime represents an asymptotically locally AdS dynamical black hole or wormhole. In three dimensions, the scalar field becomes trivial and the solution reduces to the BTZ (Bañados-Teitelboim-Zanelli) black hole.

1 Motivation and summary

The motivation of this study is twofold. Firstly to provide an exact AdS black hole which can be applied to the study of AdS/CFT duality in the dynamical context [1]. And secondly to find a possible final state of the recently-found nonlinear instability of the AdS vacuum [2].

The solution presented below may be a good model for further investigations to shed light on dynamical properties of AdS black holes. Interesting subjects are thermodynamical properties, dynamical stability, or Hawking radiation. This paper is based on [3].

2 System

We consider the Einstein-Klein-Gordon-Λ system in arbitrary $n(\geq 3)$ dimensions. The field equations are $G_{\mu\nu}+\Lambda g_{\mu\nu}=\kappa_n^2 T_{\mu\nu}$ and $\Box\phi=0$, where

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$$T_{\mu\nu} = (\nabla_\mu \phi)(\nabla_\nu \phi) - (1/2) g_{\mu\nu} (\nabla \phi)^2. \tag{1}$$

In the present paper, we consider n-dimensional warped product spacetimes $(g_{\mu\nu}, \mathscr{M}^n) \approx (g_{AB}, M^2) \times (\gamma_{ij}, K^{n-2})$ with the line element

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= g_{AB}(y) dy^A dy^B + R(y)^2 \gamma_{ij}(z) dz^i dz^j, \end{aligned} \tag{2}$$

where g_{AB} is a Lorentzian metric on M^2 and R is a scalar on M^2. K^{n-2} is an $(n-2)$-dimensional unit space of constant curvature, where k denotes its curvature taking the values 1, 0, and -1, and γ_{ij} is the metric on K^{n-2}.

The generalized Misner-Sharp quasi-local mass is a scalar on M^2 defined by

$$m := \frac{(n-2)V_{n-2}^{(k)}}{2\kappa_n^2} R^{n-3} \left(-\tilde{\Lambda} R^2 + k - (DR)^2 \right), \tag{3}$$

where $\tilde{\Lambda} := 2\Lambda/[(n-1)(n-2)]$, $(DR)^2 := g^{AB}(D_A R)(D_B R)$, and D_A is the covariant derivative on M^2 [4, 5, 6, 7]. $V_{n-2}^{(k)}$ denotes the volume of K^{n-2} if it is compact and otherwise arbitrary. m has the monotonicity and positivity properties for arbitrary (positive) $V_{n-2}^{(k)}$ and is constant in vacuum [6, 7]. In the asymptotically flat or AdS case, that coefficient is fixed in such a way that it converges to the global mass such as the Arnowitt-Deser-Misner mass [8] or Abbott-Deser mass [9].

3 Exact asymptotically locally AdS solutions

Here we particularly consider $\Lambda < 0$ and the metric in the following form:

$$ds^2 = H(\rho)^{-2} \left(-dt^2 + d\rho^2 + S(t) \gamma_{ij}(z) dz^i dz^j \right), \tag{4}$$

$$H(\rho) := \sqrt{-\tilde{\Lambda}} \sin \rho. \tag{5}$$

The domain of ρ is given by $N\pi < \rho < (N+1)\pi$ ($N \in \mathbb{Z}$) since $\rho = N\pi$ corresponds to the AdS infinity, where

$$\lim_{\rho \to N\pi} R^{\mu\nu}{}_{\rho\sigma} = \tilde{\Lambda} (\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho) \tag{6}$$

is satisfied. The function $S(t)$ is obtained by solving Einstein equations and the domain of t is determined by $S(t) > 0$. The areal radius is given by $R = (\varepsilon H)^{-1} S^{1/2}$, where $\varepsilon = \pm 1$ is chosen such that R is non-negative.

The Einstein equations require that the Klein-Gordon field is homogeneous $\phi = \phi(t)$. Then, the energy-momentum tensor has the form of $T^\mu{}_\nu = \mathrm{diag}(-\mu, \mu, \cdots, \mu)$, where $\mu := (1/2) H^2 \dot{\phi}^2$ is the energy density of the scalar field. Finally, the system reduces to the following master equation for $X(t) := S^{(n-2)/2}$:

$$E = \frac{1}{2}\dot{X}^2 + V_{(\mathrm{k})}(X), \tag{7}$$

$$V_{(\mathrm{k})}(X) := \frac{(n-2)^2}{2}\left(kX^{2(n-3)/(n-2)} + X^2\right), \tag{8}$$

where a dot denotes the derivative with respect to t and E is an integration constant.

This class of solutions has been investigated with a stiff fluid because it is equivalent to a massless Klein-Gordon field if the gradient of the scalar field is timelike. The first solutions were obtained by Lake [10] and independently obtained by other authors [11, 12, 13] in the spherical case in four dimensions ($k = 1$ and $n = 4$). The four-dimensional solutions with general k were obtained in [14, 15].

The master equation (7) is solved analytically in three and four dimensions for any k but only for $k = 0$ in higher dimensions. In four dimensions, S is given by

$$S(t) = \frac{1}{2}(-k + 2C_1 \sin 2t), \tag{9}$$

where C_1 is a constant relating to E. The energy density of the scalar field μ and the generalized Misner-Sharp mass m are given by

$$\mu = \frac{(4C_1^2 - k^2)H^2}{4\kappa_4^2 S^2}, \quad m = \frac{V_2^{(k)}(4C_1^2 - k^2)}{4\kappa_4^2 \varepsilon H S^{1/2}}. \tag{10}$$

The scalar field with positive μ (namely $4C_1^2 > k^2$) is given by

$$\pm(\phi - \phi_0) = \begin{cases} \sqrt{\frac{1}{2\kappa_4^2}} \ln\left|\frac{\sqrt{4C_1^2 - k^2} + (-k\tan t + 2C_1)}{\sqrt{4C_1^2 - k^2} - (-k\tan t + 2C_1)}\right| & [k = 1, -1], \\ \sqrt{\frac{1}{2\kappa_4^2}} \ln\left|\frac{1 - \cos 2t}{\sin 2t}\right| & [k = 0], \end{cases} \tag{11}$$

while the scalar field with negative μ is

$$\pm(\phi - \phi_0) = i\sqrt{\frac{2}{\kappa_4^2}} \arctan\left(\frac{-k\tan t + 2C_1}{\sqrt{k^2 - 4C_1^2}}\right), \tag{12}$$

where $i^2 = -1$ and ϕ_0 is a constant. In arbitrary dimensions with $k = 0$, S and ϕ are given by

$$S(t) = C_1[\sin(n-2)t]^{2/(n-2)}, \tag{13}$$

$$\pm(\phi - \phi_0) = \sqrt{\frac{n-3}{(n-2)\kappa_n^2}} \ln\left|\frac{1 - \cos(n-2)t}{\sin(n-2)t}\right|. \tag{14}$$

Even in other cases, the qualitative property of the solution is easily understood because the master equation represents a simple one-dimensional potential problem for the variable $X(t)(\geq 0)$. In general, ϕ, μ, and m are given by

$$\phi = \pm\sqrt{\frac{2(n-3)E}{(n-2)\kappa_n^2}}\int^t \frac{d\bar{t}}{S(\bar{t})^{(n-2)/2}}, \tag{15}$$

$$\mu = \frac{(n-3)EH^2}{(n-2)\kappa_n^2 S^{n-2}}, \quad m = \frac{EV_{n-2}^{(k)}}{(n-2)\kappa_n^2(\varepsilon H)^{n-3}S^{(n-3)/2}}, \tag{16}$$

where $S(t)$ is determined by the master equation (7). μ and m are positive (negative) for $E > (<)0$ and then the scalar field is real (pure imaginary, namely ghost). In three dimensions ($n = 3$), the scalar field becomes trivial and we have $\mu = 0$ and m =constant. The spacetime is then locally (A)dS and represents the BTZ (Bañados-Teitelboim-Zanelli) black hole in the non-standard coordinates [16, 17].

Equation (16) shows that the spacetime is vacuum at AdS infinity ($H = 0$), but m blows up there. This means that the fall-off rate to the AdS infinity is slower than the Henneaux-Teitelboim condition and hence the spacetime is asymptotically AdS only locally.

4 Physical interpretations

We clarify the causal structure of the spacetime (4) and give its physical interpretations. For this purpose, the following three facts are important; (i) $\rho = N\pi$ ($N \in \mathbb{Z}$) is AdS infinity, which is timelike, (ii) $S(t) = 0$ corresponds to curvature singularity, which is spacelike, and (iii) a light ray runs along a 45-degree straight line in the (ρ,t)-plane since the metric on M^2 in the solution (4) is conformally flat.

The present coordinate system covers the maximally extended spacetime and the Penrose diagrams for the spacetime (4) are presented in Fig. 1. (See Table 1.) The spacetime represents a dynamical black hole or wormhole depending on the parameters.

Table 1 The Penrose diagrams of the solution with positive energy density and $n \geq 4$. In the case of $k = -1$ with negative energy density, the Penrose diagram is Fig. 1(d).

	$n = 4$	$n \geq 5$
$k = 0$	Fig. 1(b)	Fig. 1(c)
$k = 1$	Fig. 1(c)	Fig. 1(a), (b), or (c)
$k = -1$	Fig. 1(a)	Fig. 1(a), (b), or (c)

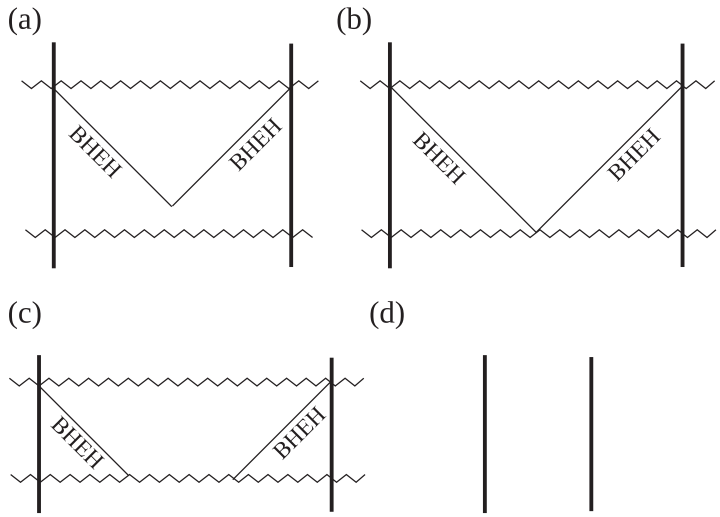

Fig. 1 The Penrose diagrams for the solution. A zigzag and a thick line correspond to a curvature singularity and AdS infinity, respectively. BHEH stands for the black-hole event horizon. Figs. (a)–(c) represent a black hole, while Fig. (d) represents a wormhole.

4.1 Dynamical AdS black holes

If the scalar field is real (and equivalently the energy density is positive), there are spacelike curvature singularities given by $S(t) = 0$ in the (ρ,t)-plane. As a result, the solution represents a dynamical black hole. Since both H and S are periodic, the (ρ,t) plane is divided by singularities and AdS infinities ($\rho = N\pi$) into an infinite number of portions. All the portions with positive S are equivalent.

First let us see the case with $k = 0$. Without loss of generality, we assume $C_1 > 0$ in (13) and consider a physical portion defined by $t = (0,\pi/(n-2))$ and $\rho = (0,\pi)$, which covers the maximally extended spacetime. The event horizon in this portion is given by $t = \rho - (n-3)/(n-2)\pi$ and $t = -\rho + \pi/(n-2)$ and the Penrose diagram is (b) in Fig. 1 for $n = 4$ and (c) for $n \geq 5$.

On the other hand, in the case with $k = 1,-1$ in four dimensions, the period of t in a physical portion is different. The period is shorter (longer) than $\pi/2$ for $k = 1$ $(k = -1)$. Hence, the Penrose diagram is (c) in Fig. 1 for $k = 1$ and (a) for $k = -1$.

In the case with $k = \pm 1$ and $n \geq 5$, the solution is not obtained in a closed form, but we can prove that it represents an AdS black hole if the energy density of the scalar field is positive, namely $E > 0$. For $k = 1$, the potential (8) is monotonically increasing for $X \geq 0$ and hence the solution exists only for $E > 0$. Then, the domain of t in the maximally extended spacetime of the solution is given by $t_0 < t < t_0 + T$, where $X(t_0) = X(t_0 + T) = 0$. This is also the case for $k = -1$ with $E \geq 0$. The period T is defined by

$$T := 2\int_{X=0}^{X=X_{\rm b(k)}} \frac{dX}{\sqrt{2(E-V_{\rm (k)}(X))}}, \tag{17}$$

where $X_{\rm b(k)}$ is defined by $E = V_{\rm (k)}(X_{\rm b(k)})$. The spacetime admits a wormhole throat if $T \geq \pi$ because the period of the coordinate ρ is π, however it is not allowed if the scalar field has positive energy density. (See Appendix C in [3] for the proof.) Since $t = t_0$ and $t = t_0 + T$ are both spacelike curvature singularities, the corresponding Penrose diagram is Figs. 1(a), 1(b), and 1(c) for $\pi/2 < T < \pi$, $T = \pi/2$, and $0 < T < \pi/2$, respectively. Although the diagrams are different depending on the value of T, the solution represents a dynamical AdS black hole.

4.2 Dynamical AdS wormholes

In the case of $k = -1$ in four dimensions, if $4C_1^2 < k^2$, then the energy density is negative and S is positive definite for $-\infty < t < \infty$. (There is no physical solution for $k = 1$ because S is negative definite if $4C_1^2 < k^2$.) The Klein-Gordon field then becomes ghost and there is no curvature singularity in the spacetime. As a result, the spacetime is a dynamical AdS wormhole described by the Penrose diagram (d) in Fig. 1.

It is shown that an AdS wormhole is realized also for $k = -1$ and $n \geq 5$ if $E < 0$; namely the energy density is negative (and equivalently the scalar field is ghost). In the case of $k = -1$, the potential (8) in the master equation has a negative minimum $V_{(-1)} = V_{\rm ex}$, where

$$V_{\rm ex} := -\frac{n-2}{2}\left(\frac{n-3}{n-2}\right)^{n-3} (< 0). \tag{18}$$

As a result, for the solution with E satisfying $V_{\rm ex} < E < 0$, the value of X (and hence S) oscillates and never becomes 0. Hence, the corresponding Penrose diagram is Fig. 1(d) and the spacetime describes a dynamical AdS wormhole.

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