Gravomagnetic Solenoids

Donald Lynden-Bell and Joseph Katz

Abstract We introduce strong field gravomagnetism and illustrate its use by constructing exact rolling toroidal solutions of Einstein's equations.

1 Introduction

The 1966 edition of the Classical theory of Fields by Landau and Lifshitz [1] gives Einstein's equations for general stationery metrics in a form that has strong analogies with Maxwell's electrodynamics. The technique identifies the points of space that lie along the time-like Killing vector, so it does not extend continuously inside ergospheres where the Killing vector becomes space-like. We write the metric in the form

$$ds^2 = \xi^2 (dt - \mathscr{A}_k dx^k)^2 - \gamma_{kl} dx^k dx^l = g_{\mu\nu} dx^\mu dx^\nu. \tag{1}$$

We work in the positive definite three dimensional metric of space, γ_{kl}, where k and l run from 1 to 3. It is not a cross-section of the four metric by any surface, nevertheless we may define its Christoffel symbols λ^m_{kl} and the corresponding three-dimensional Ricci tensor of this gamma space, P^{kl}. We use commas to denote ordinary derivatives and semicolons to denote covariant derivatives in gamma space. The Ricci tensor of space-time will be denoted by $R_{\mu\nu}$. The divergence and curl are defined in gamma-space by

$$\mathbf{div}\,\boldsymbol{E} = \gamma^{-\frac{1}{2}}\,\partial_k(\sqrt{\gamma}E^k) \ \ ; \ \ (\mathbf{curl}\,\boldsymbol{E})^i = \gamma^{-\frac{1}{2}}\,\varepsilon^{ijk}\partial_j E_k. \tag{2}$$

We define the gravomagnetic induction $\mathscr{B}$ by

Donald Lynden-Bell and Joseph Katz
Institute of Astronomy, Madingley Rd, Cambridge, England, e-mail: dlb@ast.cam.ac.uk

Joseph Katz
The Racah Institute of Physics, Edmond Safra Campus, Givat Ram, 91904 Jerusalem, Israel

$$\mathscr{B} = \textbf{curl}\ \mathscr{A}, \tag{3}$$

where $\mathscr{A}_k$ is the vector potential defined in the metric (1). Clearly $\textbf{div}\ \mathscr{B} = 0$ so $\mathscr{B}$ carries the gravomagnetic flux. Landau and Lifshitz rewrite Einstein's equations in gamma space; rewriting their equations in our notation we have with $\kappa = 8\pi G/c^4$,

$$\xi\, \textbf{div}\, \textbf{grad}\, \xi + \frac{1}{2}\ \xi^4 \mathscr{B}^2 = R_{00} = \kappa(T_{00} - \frac{1}{2}\ g_{00}T). \tag{4}$$

Henceforth we use units with $c = 1$ and $G = 1$. If we now define a field intensity vector $\mathscr{H} = \xi^3 \mathscr{B}$ then their second equation reads

$$(\textbf{curl}\ \mathscr{H})^k = -2\kappa\xi T_0^k = -2\kappa J^k. \tag{5}$$

Notice a strong resemblance of this strong field equation to Maxwell's electrodynamic equation $\textbf{curl}\,\mathbf{H} = 4\pi\mathbf{j}$. In both cases the current has no divergence however in general $\mathscr{H}$ has a divergence while $\mathscr{B}$ does not. Clearly $\mathscr{H}$ is the gradient of a scalar whenever $\mathbf{J}$ is zero. The $\mathscr{H}^k$ are the spatial components of the twist vector $\eta^{\mu\nu\sigma\tau}\xi_\nu D_\tau \xi_\sigma$ where D is the covariant derivative in the space-time $g_{\mu\nu}$. The last Einstein equation is

$$P^{kl} + \frac{1}{2}\ \xi^2(\gamma^{kl}\mathscr{B}^2 - \mathscr{B}^k\mathscr{B}^l) - \xi^{-1}\xi^{;k;l} = R^{kl} = \kappa(T^{kl} - \frac{1}{2}\ g^{kl}T). \tag{6}$$

To illustrate the power of the analogy between electromagnetism and gravomagnetism we consider first a straight solenoid of length L with a wire wound n times around it carrying a current I; the magnetic flux inside is $F = 4\pi nI$. This flux emerges from the pole at one end of the solenoid and spreads over an area of order L^2 before returning to the pole at the other end. Even if n increases linearly with the the length L the equatorial field outside the solenoid F/L^2 decreases like $1/L$ so, as the length of the solenoid increases, the field outside tends to zero.
The analogous gravitational case is a rotating cylindrical shell and the case that is solved in General Relativity is the infinitely long cylinder. In such a case the external gravomagnetic field will be zero as the returning flux is sent to infinite distances. Thus it is not surprising that the external metric is static rather than stationary.

Our second, less trivial problem is the toroidal solenoid for which the electrical case is illustrated below. Here the magnetic field is confined within the solenoid and there are no poles from which it emerges. Indeed even if the torus carries a charge as well as a current the external field is purely electrical without any magnetism. The analogous gravitational problem is a massive toroidal shell that rolls around the circumference of its small cross-section so that the equator closest to the global axis moves up while the equator furthest from the axis moves down. We find exact solutions to this problem in General Relativity in which the metric is static outside the torus but stationary and non-static inside the torus.

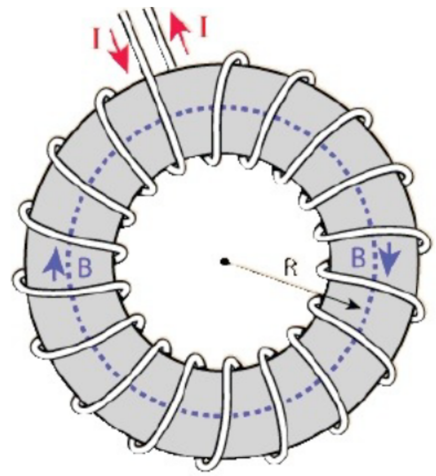

Fig. 1 In electromagnetism a wire carrying a current I tightly wound n times around a torus produces a field confined within the Torus. In the mathematics the wire is replaced by a continuum whose total current is called I (not nI)

2 The general static Weyl metric in toroidal coordinates

Weyl takes the metric in the form

$$ds^2 = e^{-2\psi}dt^2 - e^{2\psi}\left[e^{2k}\left(dz^2+dR^2\right)+R^2d\varphi^2\right] \tag{7}$$

Then, in empty axially symmetric spaces Einstein's equations give $\nabla^2\psi = 0$ where ∇^2 is the flat space operator. Also setting

$$D = \partial_R - i\partial_z, \tag{8}$$

we have the Weyl equations

$$Dk = \tfrac{1}{4}Re^{4\psi}De^{-2\psi}De^{-2\psi}. \quad \text{So} \quad DkD\ln R = (D\psi)^2. \tag{9}$$

The general symmetric solution to these equations in toroidal coordinates is given by [2] whose solution does not include that for k given below,

$$\psi = \sqrt{u-v}\,U, \quad \text{where} \quad U = \sum_{l=0}^{\infty} a_l P_L(u)\cos(l\eta). \tag{10}$$

Here $P_L(u)$ is the Legendre function, $L = l+\frac{1}{2}$; $u = \cosh\zeta$, $v = \cos\eta$,

$$R = h\sinh\zeta \quad , \quad z = h\sin\eta \quad \text{with} \quad h = \frac{a}{\cosh\zeta - \cos\eta}. \tag{11}$$

We then have

$$dR^2 + dz^2 = h^2\left(d\zeta^2 + d\eta^2\right). \tag{12}$$

Thus h and R are the scale factors in toroidal coordinates. a is the radius of the central line torus. On each given torus ζ is constant. On the axis $R=0$, $\zeta=0$ and ζ is also zero at infinity. On $z=0$, $\eta=\pi$ when $R<a$ and $\eta=0$ when $R>a$. The function k is given by,

$$k = \frac{1}{8}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_l a_m \left[c(l+m+1)k^1_{l,m} + c(l+m)k^0_{l,m} + c(l+m-1)k^{-1}_{l,m}\right]$$
$$+\frac{1}{8}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_l a_m \left[c(l-m+1)k^1_{l,-m} + c(l-m)k^0_{l,-m} + c(l-m-1)k^{-1}_{l,-m}\right], \qquad (13)$$

where $k^n_{l,m}$ are known terms of Legendre functions themselves and their derivatives and $c(m)$ is short for $\cos(m\eta)$.

3 A toroidal solenoid's metric and junction conditions

Inside our torus we need a solution with a toroidal gravomagnetic field. For this we take Bonnor's nice solution [3] for the metric external to a light beam characterised as null dust. This clearly must have a toroidal gravomagnetic field. We generalise the metric by incorporating a constant conicity $\bar{k}$. Since the symmetry axis is no longer included in the part of the solution used within our torus, this does not generate a singularity and is necessary to accomplish the fitting to the external metric. Bonnor's metric is then,

$$ds^2 = F[dt-(1-F^{-1})d\bar{z}]^2 - (e^{2\bar{k}}d\bar{R}^2 + \bar{R}^2 d\varphi^2 + F^{-1}d\bar{z}^2), \qquad (14)$$

where $F = 8I\,\ln(\bar{R}/a) + \bar{C}$, $\bar{C}$ is a constant and I the total current that generates the gravomagnetic field; in Bonnor's application it is the current caused by the light ray; in our application it is the total current that runs around our torus by the short way. Our task is now reduced to fitting this metric for the inside of our torus to our general Weyl metric for the outside. Before attempting this full problem we solved the easy case of a static equipotential shell with a Bach-Weyl [4] exterior metric. The Riemann tensor inside the equipotential toroid is zero but the metric there though locally flat is actually "conical", so it is not globally flat. $\bar{k}$ is a negative constant not zero. It was solving this simpler problem that alerted us to the necessity of incorporating such a conicity into Bonnor's metric in the general problem. The shapes of the Bach-Weyl equipotential surfaces are shown below. Inside the limiting one that intersects the axis, the conicity, $\bar{k}$, is zero so for it the internal space is globally flat. A massive shell may be placed on any one of these equipotential surfaces leaving the shapes of the external equipotentials unchanged but of course changing the values of their potentials. Inside such a shell the potential is constant. We determined the limits to these masses so that the pressure components in the shell do not exceed the limits set by the energy conditions.

For our rolling toroid the complete set of Einstein's equations are given in equations

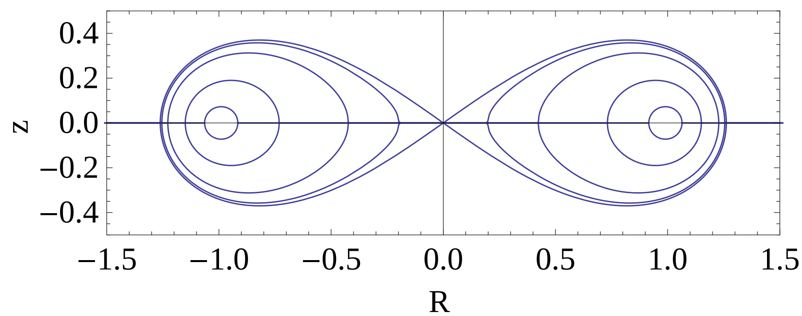

Fig. 2 Equipotentials of the Bach-Weyl static Toroids.

(4), (5) and (6). None of these equations involve $\mathscr{A}$ itself as opposed to $\mathscr{B} = \mathbf{curl}\mathscr{A}$, so the boundary conditions for all such problems can be expressed in terms of $\mathscr{B}$ rather than $\mathscr{A}$. Four of them are strikingly similar to the boundary conditions in electrodynamics. These are: that the potential ψ is continuous, that $\mathscr{B}.\boldsymbol{n}$ is continuous, that $\mathscr{H} \times \boldsymbol{n} = 16\pi\mathbf{J}$ (the gravomagnetic field being zero outside) and that from integrating (1.4) the discontinuity in the potential gradient along the normal is related to an appropriate surface density. We also need the gamma metrics on the surfaces to be identical and from integrating equation (6) across the surface we require the discontinuities of the external curvatures of the gamma space 3-metrics to be related to the stresses in the toroid. While this procedure is equivalent to Israel's it is easier to carry out since it does not involve $\mathscr{A}$ and fits the surfaces in space rather than space-time. We give below an example of one of our strongly relativistic rolling toroids for which we carried out this procedure to ensure that the surface stresses obeyed the dominant energy conditions.

4 A strongly relativistic example

The dominant energy condition limits how much mass our torus can withstand at a given rolling rate.
Equatorial radii $a_{in}/a = 0.826$: $a_{out}/a = 1.21$. Axial ratio $b/a = 0.297$.
Total mass $M/a = 0.659$. Matter current $I = 0.008$
Rolling speeds across equators $v_{in,\ out}/c = 0.85,\ 0.09$.
Equatorial pressures $(p_\phi/\sigma)_{in,out} = 0.28,\ 0.934; (p_\eta/\sigma)_{in,out} = 0.1,\ -0.57$.
We tested our calculations by showing that in the non-relativistic limit the stresses calculated by the relativistic method balanced the gravity and the centrifugal force due to rolling. More details of this work are given in [5].

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