

# On the Effects of Rotating Gravitational Waves

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**Abstract** We study effects of gravitational waves which in the first order form a time-symmetric ingoing and then outgoing pulse of rotating waves. The influence of the angular momentum of these waves on the rotation of local inertial frames with respect to the frames at great distances is analyzed by solving the relevant Einstein equation to second order. Also the apparent motions of the fixed stars on the celestial sphere as seen through rotating waves from the local inertial frame in the centre are calculated and displayed.

## 1 Introduction

It was just 100 years ago in Prague when Einstein wrote the paper [1] in which he, for the first time, expressed his understanding of Mach's Principle. Within his pre-General Relativity theory in which there was only one metric function he considered a mass point inside a shell accelerated "upwards" and found that the mass-point is dragged along by the shell.

Many formulations and studies of Mach's Principle appeared during the last 100 years, most of them were analyzed in the Tübingen conference in 1993 which led to the remarkable volume [2] containing lectures as well as valuable discussions. We studied Machian effects in various contexts, both in asymptotically flat space-

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times and within cosmological perturbation theory – see, e.g., [3], and number of references therein; later, cf. Schmid [4].

More recently, we investigated a subtle question whether dragging of inertial frames should be attributed also to gravitational waves. After the discovery of binary pulsars losing energy and angular momentum as a consequence of emitting gravitational radiation it would be surprising if gravitational waves did not have an influence on local inertial frames. However, there are still doubts uttered about the status of gravitational stress-energy as compared with stress-energy tensor  $T_{\mu\nu}$  of matter in relation to Machian ideas (see, e.g., [2], p. 83).

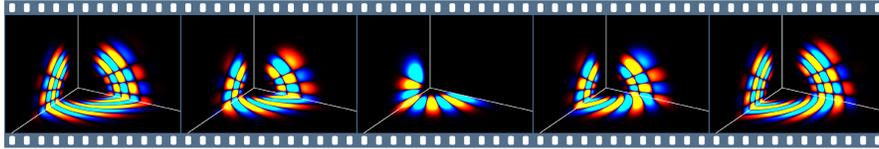
In the present work [5] we investigate the effects of rotating gravitational waves in a more general, asymptotically flat setting, without assuming cylindrical symmetry. We again start out from linearized theory and construct an ingoing rotating pulse of radiation which later transforms into an outgoing pulse. While in the cylindrical case our waves were characterized by just one harmonic index  $m$  governing the number of wave crests in  $\varphi$ , now the situation becomes considerably richer involving both spherical harmonic indices  $l$  and  $m$ .

Near the origin the first-order metric of our waves behaves as  $r^l$ , so the region around the origin will be very nearly flat for  $l$  sufficiently large. When, however, a local inertial frame is introduced at the origin, we find that its axes rotate with the angular velocity  $\omega_0$  with respect to the lines  $\varphi = \text{const}$  of the global frame, i.e., with respect to stars at infinity. The situation thus indeed resembles the interior of a collapsing slowly rotating shell - see [6] where the vorticity of the lines  $\varphi = \text{const}$  is given in covariant form.

## 2 Rotating scalar and gravitational waves

We first construct a solution of the scalar wave equation in a form of the rotating wave pulse written in spherical coordinates  $t, r, \theta, \phi$

$$\psi_{lm}(t, r, \theta, \phi) = Q_l(t, r) Y_{lm}(\theta, \phi) = B_l 2^l l! \frac{(r/a)^l Y_{lm}(\theta, \phi)}{[(a+it)^2 + r^2/a^2]^{l+1}}, \quad (1)$$



**Fig. 1** The snapshots of the rotating waves  $\psi_{lm}$  in times  $t/a = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$  (from left to right) for  $l = 17, m = 11$ . Waves are localized in the radial direction so that they resemble a falling and rotating shell.

where  $l, m$  are harmonic indices,  $a$  is typical wave pulse width and  $B_l$  is the amplitude. Here both numerator and denominator are complex functions so the actual wave profile given by its real part has plenty of features as can be seen in Figure 1. Among them the rotating character of the wave, its regularity and the fact that for high values of  $l$  the wave is concentrated near a shell with radius  $r^2 = a^2 + t^2$  are most important.

In the construction of rotating gravitational waves within linearized Einstein theory we then use the Regge-Wheeler equation for odd-parity waves [7] which on the flat Minkowski background simplifies to a usual wave equation  $\square\psi = 0$ . We decompose the metric perturbations into tensor harmonics [7], [8], [9] and consider only the odd-parity waves with the Regge-Wheeler gauge condition ( $(i) = (1), (2)$  denotes the first- and second-order perturbations)

$$h_{\mu\nu}^{(i)} = \sum_{lm} \frac{\sqrt{2l(l+1)}}{r} \left[ -h_{0lm}^{(i)}(t, r) c_{0lm\mu\nu} + ih_{1lm}^{(i)}(t, r) c_{lm\mu\nu} \right] \quad (2)$$

where  $c_{0lm}, c_{lm}$  are the odd-parity harmonics [9]. The first order radial functions  $h_{0lm}^{(1)}(t, r) = -\partial_r(r^2 Q_l)/(l^2 + l - 2)$  and  $h_{1lm}^{(1)}(t, r) = -\partial_t(r^2 Q_l)/(l^2 + l - 2)$  of the odd-parity metric perturbations we directly obtain from the radial part  $Q_l(t, r)$  of the scalar field  $\psi_{lm}$ .

In [5] we also analyze and relate the energy and angular momentum densities of scalar and gravitational waves.

### 3 Second order perturbations

To determine the influence of gravitational waves  $h_{\mu\nu}^{(1)}$  on the rotation of local inertial frames at the axis of symmetry due to the second-order metric perturbations  $h_{\mu\nu}^{(2)}$  we solve the equations

$$R_{\mu\nu}^{(1)}[h^{(2)}] = - \langle R_{\mu\nu}^{(2)}[h^{(1)}, h^{(1)}] \rangle. \quad (3)$$

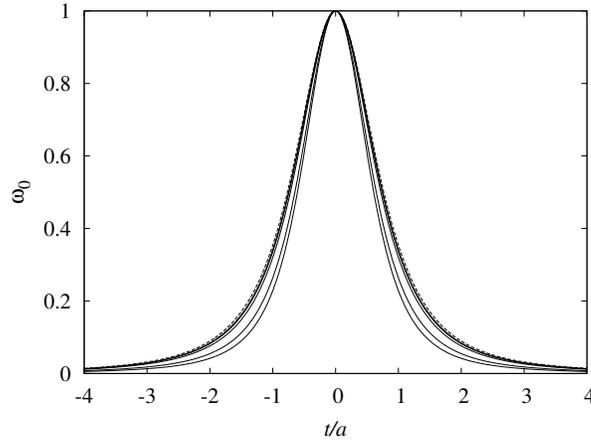
where we introduced the averaging symbol  $\langle \rangle$ . We expand both sides in tensor spherical harmonics. For general  $l$  the l.h.s. yields a hyperbolic set of equations for radial functions  $h_0^{(2)}$  and  $h_1^{(2)}$  indicating non-instantaneous effects, but the inertial frames at the origin will be influenced primarily by the *dipole* perturbations and it is well known that for  $l = 1$  one can fix  $h_1^{(2)} = 0$  by an appropriate gauge transformation [7], [10] and arrive thus at *elliptic* equation for  $h_0^{(2)}$ . In this equation the axially symmetric component  $\langle R_{r\phi}^{(2)}[h^{(1)}, h^{(1)}] \rangle$  on the r.h.s. of (3) appears as the only source for the dipole second-order perturbations

$$\partial_{rr} h_0^{(2)}(t, r) - \frac{2}{r^2} h_0^{(2)}(t, r) = \frac{4\pi}{\sqrt{2l(l+1)}r} \int_0^\pi \langle R_{r\phi}^{(2)} \rangle \partial_\theta Y_{l0} d\theta. \quad (4)$$

This equation can be solved by variation of constants. While the Coriolis and centrifugal accelerations are higher order in the angular velocity  $\omega_0$  of the rotation of an inertial frame (of a gyroscope) located near the origin,  $g_{t\varphi}^{(2)} = -\omega_0 r^2 \sin^2 \theta$  there and so the behavior of  $h_0^{(2)}$  at  $r = 0$  determines

$$\omega_0 = \frac{1}{4\pi} \int_0^\infty \int_{S^2} R_{t\varphi}^{(2)}[h^{(1)}, h^{(1)}] d\Omega \frac{dr}{r}. \quad (5)$$

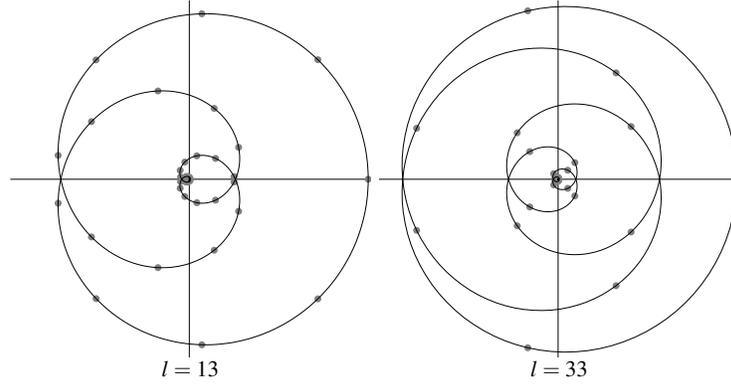
Although  $R_{t\varphi}^{(2)}[h^{(1)}, h^{(1)}]$  has a complicated structure, we obtained the angular velocity  $\omega_0$  in the closed, although quite lengthy, form. The profiles of  $\omega_0(t)$  are in Figure 2.



**Fig. 2** The dependence of normalized angular velocity of the central inertial frame  $\omega_0(l, 1; t)/\omega_0(l, 1; t = 0)$  on the parameter  $l = 2, 3, 10, 20, 30$  (from inside to out). The dashed line indicates the limit for large  $l$ .

## 4 Observing stars through gravitational waves

We evaluate the first-order effects of the waves on the propagation of photons which apparently change the position of distant stars in the sky as seen by an observer fixed in the flat region at the origin. We found that the change of apparent star's celestial coordinates  $\delta\varphi, \delta\theta$  can be computed as a perturbation of the ingoing radial null geodesic. This change of the direction of a momentum of the photon registered at time  $T$  can be written using integrals along the unperturbed ray. They yield quite simple formula for trajectory of the star with initial coordinates  $\varphi, \theta$  in a form of conformal mapping of a straight line in a complex plane by function  $z^{-l-2}$



**Fig. 3** Since light from distant stars is influenced by the gravitational waves the observed positions of the stars change. An observer at the origin can record the apparent position of the stars on the celestial sphere on a photographic plate. When appropriately scaled and rotated, the trajectories of all stars are the same. A star starts at the origin of the plate ( $x = y = 0$  in the planes above) and moves along closed trajectories the structure of which becomes more complicated with increasing  $l$ . The trajectory of a star with celestial longitude  $\varphi$  fits in an ellipse with semi-axes  $\Delta\theta, \Delta\varphi$  rotated on the celestial sphere by the angle  $l\pi/2 + m\varphi$  (see Eq. (6)).

$$\frac{\delta\varphi(T)}{\Delta\varphi} + i\frac{\delta\theta(T)}{\Delta\theta} = \frac{i^l e^{im\varphi}}{(1 + i\frac{T}{a})^{l+2}}. \quad (6)$$

where  $\Delta\theta = \hat{B}_l m P_l^m(\cos\theta)/\sin\theta$ ,  $\Delta\varphi = -\hat{B}_l P_l^m(\cos\theta)$ , and  $\hat{B}_l = N_l^{m2}(l-1)!$ . See Figure 3 for examples of star image trajectories.

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