The Null Geodesics in the Black Saturn Spacetime

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Abstract We calculate numerically null geodesics in the Black Saturn spacetime. Our analysis is restricted to the rotation axis in the domain of outer communication. The geodesics are calculated for two different Black Saturn configurations with the same ADM mass and angular momentum.

1 Introduction

In 2007 Henriette Elvang and Pau Figueras presented the single spinning, uncharged Black Saturn metric, which can be defined as "a Black Ring balanced by rotation around a concentric spherical black hole in an asymptotically flat spacetime" [1]. This solution was constructed by inverse scattering method. It is an exact, stationary, asymptotically flat 4+1 dimensional vacuum solution of Einstein's equations, where angular momentum keeps the configuration in equilibrium. This solution is really interesting, because of the 2- fold continuous non-uniqueness for fixed ADM mass and ADM angular momentum. The configuration with zero angular momentum measured at infinity makes the Schwarzschild-Tangherlini solution non-unique. The solution permits two rotational planes and also the possibility of charged Black Saturn, what is not investigated here. Such a complicated solution does not allow us to check algebraically if Einstein's vacuum equations are satisfied, but in [1] authors describe numerical tests, which show that all components of the Ricci tensor vanish.

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1.1 The metric

The metric is rather complicated, and it might be written as follows [1]

$$
\begin{aligned}
ds^2 = & -\frac{H_y}{H_x}\left[dt + \left(\frac{\omega_\psi}{H_y} + q\right) d\psi\right]^2 \\
& + H_x \left[k^2 P \left(d\rho^2 + dz^2\right) + \frac{G_y}{H_y} d\psi^2 + \frac{G_x}{H_x} d\phi^2\right]
\end{aligned}
\tag{1}
$$

where H, G, ω, P are functions of ρ and z coordinates, q is a constant included in order to ensure asymptotic flatness and k is an integration constant. The explicit formula of these functions can be found in [1].

2 Geodesics

If the space-time possesses n Killing vectors ξ_μ: $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$ there are n conserved quantities connected with these vectors. The conjugate momenta are given by $p_\mu = g_{\mu\nu} \frac{dx^\nu}{d\lambda}$ where $\mu, \nu = 1, 2, \ldots, d$, for d dimensional space-time. If the Killing field ∂_μ exists, p_μ is conserved and can be used to simplify the equations. Another equation is obtained from the null condition $g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$. All that allows us to determine the path of photons for the given metric. From the shape of the metric it is clear that there are three Killing vectors $\partial_t, \partial_\psi, \partial_\phi$ which generate three conserved quantities along the geodesics.

$$
-e = g_{tt} \frac{dt}{d\lambda} + g_{t\psi} \frac{d\psi}{d\lambda} = -\frac{H_y}{H_x} \frac{dt}{d\lambda} - \frac{H_y q + \omega_\psi}{H_x} \frac{d\psi}{d\lambda}, \tag{2}
$$

$$
\begin{aligned}
l_1 = & \; g_{\psi\psi} \frac{d\psi}{d\lambda} + g_{t\psi} \frac{dt}{d\lambda} = \\
& = -\frac{H_y q + \omega_\psi}{H_x} \frac{dt}{d\lambda} + \left(\frac{G_y H_x}{H_y} - \frac{H_y \left(q + \frac{\omega_\psi}{H_y}\right)}{H_x}\right) \frac{d\psi}{d\lambda},
\end{aligned}
\tag{3}
$$

$$
l_2 = g_{\phi\phi} \frac{d\phi}{d\lambda} = G_x \frac{d\phi}{d\lambda}. \tag{4}
$$

We set $l_1 = l_2 = 0$, which means that the angular momenta are equal to zero, e might be interpreted as just the scale of affine parameter and is chosen to be equal to one. Now, taking Taylor expansion of the metric functions we calculate them in the limit of $\rho \to 0$ and take into account only the leading terms. These calculations are based on the code used in [2]. The null condition takes the form (assuming $d\rho/d\lambda = 0$)

$$
\frac{e^2(-G_y H_x^2 + (H_y q + \omega_\psi)^2) + G_y H_x^2 H_y k^2 P \left(\frac{dz}{d\lambda}\right)^2}{G_y H_x H_y} = 0. \tag{5}
$$

3 The results

We have considered the null geodesic equations on the axis of rotation, in the region of outer communication, reduced them to the simplest form and then solved numerically in a particular case. In one of the parameterizations, the Black Saturn is defined by free parameters a_5, a_4, a_3, which are related to the position and shape of the horizons of the central black hole and the surrounding black ring [1]. In this paper, two different sets of these parameters are investigated. It is impossible to calculate the geodesics substituting the $\rho = 0$ into the metric functions, because this limit is non-trivial.

The metric functions are analytic near $\rho = 0$ in ρ for $z > a_2$. Thus, we use Taylor expansion in ρ, about $\rho = 0$. In this work we present the numerical solutions for two Black Saturn configurations, such that the ADM mass M and ADM angular momentum J are the same. If we solve the equations (3.30) and (3.31) from [1], for fixed values of ADM mass $M = 1$ and angular momentum $J = 0.5$, it is now possible to calculate the a_3 and a_5 parameters for two $a_4 = \frac{1}{2}, \frac{1}{4}$ parameter values. The following sets of parameters and Black Saturn configurations were investigated:

1. $a_5 = 0.180344, a_4 = \frac{1}{4}, a_3 = 0.81897$;
2. $a_5 = 0.45903, a_4 = \frac{1}{2}, a_3 = 0.85904$.

Using Wolfram Mathematica, the equations for $t(\lambda)$ and $z(\lambda)$ were numerically solved for this set of parameters, and the results are presented below. The functions $t(\lambda)$ seem to be the same, see figure 1 (left panel), but the difference is visible in a plot of $t_1(\lambda) - t_2(\lambda)$ see figure 1 (right panel). Almost the same situation occurs

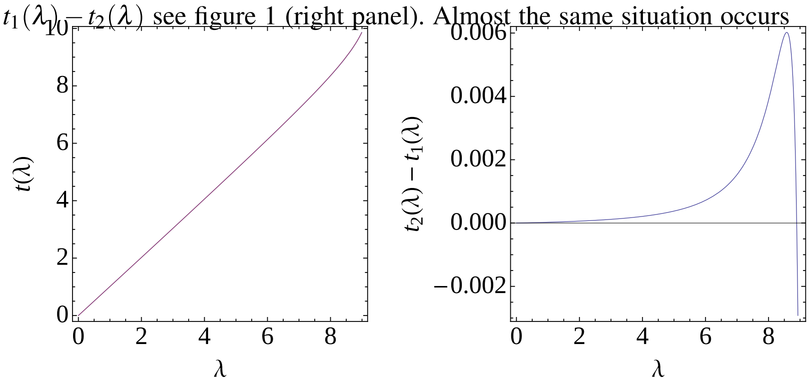

Fig. 1 Left panel - function $t(\lambda)$ for different Black Saturn configurations and the same ADM mass $M = 1$ and ADM angular momentum $J = 0.5$. **Right panel** - plot of $t_2(\lambda) - t_1(\lambda)$ difference between $t(\lambda)$ functions.

for $z(\lambda)$ functions, see figure 2 (left panel) and 2 (right panel). The figure 3 (left panel) shows the $t(z)$ dependence, where the t coordinate became the proper time for a distant, stationary observer. For increasing t, z tends to $a_2 = 1$, and it is now visible, that there is a horizon of the central black hole.

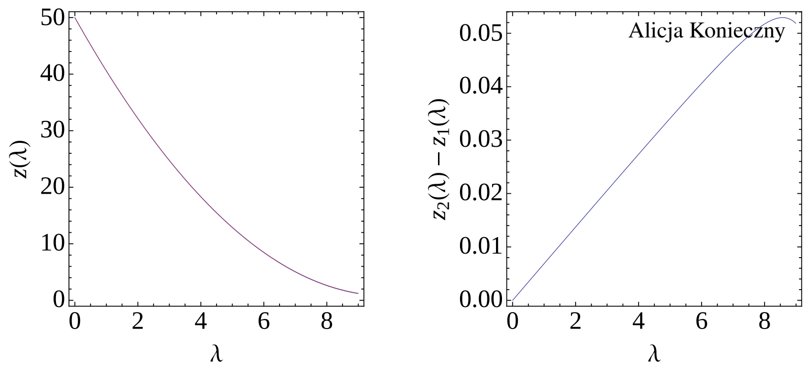

Fig. 2 **Left panel** - Function $z(\lambda)$ for different Black Saturn configurations and the same ADM mass $M = 1$ and ADM angular momentum $J = 0.5$. **Right panel** - plot of $z_2(\lambda) - z_1(\lambda)$ difference between $z(\lambda)$ functions.

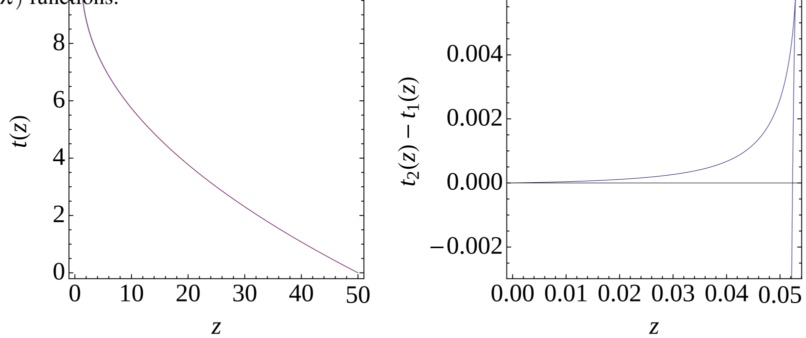

Fig. 3 **Left panel** - The parametric plot of $t(z)$ geodesics for fixed Black Saturn ADM mass $M = 1$ and ADM angular momentum $J = 0.5$. **Right panel** - plot of difference $t_2(z_2) - t_1(z_1)$ of $t(z)$.

References

1. H. Elvang, P. Figueras, *Black Saturn*, J. High Energy Phys. **2007**(05), 050 (2007)
2. P.T. Chruściel, M. Eckstein, S.J. Szybka, *On smoothness of black saturns*, J. High Energy Phys. **2010**(11), 048 (2010)