Averaging Inside the LRS Family

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Abstract Averaging problem in GR and cosmology is of fundamental importance. It is still not clear how to unambiguously average Einstein equations and the metric tensor. One of the most promising attempts how to deal with averaging in GR are the Buchert equations. However, only scalar part of the Einstein equations is averaged and the system is not closed. Here we will present LRS (locally rotationally symmetric) spacetimes, where one can find preferred spatial direction and the evolution and the constraint equations are described only by scalars. By averaging these scalars we will obtain generalized Buchert equations (for LRS spacetimes).

1 Introduction

The averaging problem in general relativity was studied by Ellis [1] and has been investigated many times since then. However despite some attempts [2] it is not obvious how to take an average of a tensorial quantity. On the other hand averaging of scalars according to Buchert [3] is a fully covariant operation, but it has some drawbacks as well. It is performed over some domain on a spatial hypersurface and it depends on the slicing of the spacetime and of course on the scale.

The key problem in cosmology is that calculating the Einstein tensor from the averaged metric is not the same thing as calculating the Einstein tensor from inhomogeneous metric and taking the average after that

$$G_{\mu\nu}\left(\langle g_{\mu\nu}\rangle\right) \neq \langle G_{\mu\nu}\left(g_{\mu\nu}\right)\rangle . \tag{1}$$

This property follows from the nonlinearity of Einstein equations.

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2 Buchert equations

Averaging of scalars was derived by Buchert [3]. The average of a scalar ψ over a domain $\mathscr{D}$ on a spatial hypersurface is defined as

$$\langle\psi\rangle \equiv \frac{1}{V_{\mathscr{D}}} \int_{\mathscr{D}} \psi \sqrt{\det g_{ij}}\, \mathrm{d}^3 x, \tag{2}$$

where $V_{\mathscr{D}}$ is volume of the domain $\mathscr{D}$. Taking the average of the Raychaudhuri equation and the Hamiltonian constraint leads to Buchert equations

$$3\frac{\ddot{a}_{\mathscr{D}}}{a_{\mathscr{D}}} = -4\pi\langle\rho\rangle + \mathscr{Q}, \tag{3}$$

$$3\frac{\dot{a_{\mathscr{D}}}^2}{a_{\mathscr{D}}^2} = 8\pi\langle\rho\rangle - \frac{1}{2}\langle\mathscr{R}\rangle - \frac{1}{2}\mathscr{Q}. \tag{4}$$

Here ρ is the matter density, $\mathscr{R}$ is the Ricci scalar on the spatial hypersurface and $a_{\mathscr{D}}$ is the effective scale factor of the domain $\mathscr{D}$. The quantity $\mathscr{Q}$ in Buchert equations (3) - (4) is called the backreaction and is defined as

$$\mathscr{Q} \equiv \frac{2}{3}\left(\left\langle\theta^2\right\rangle - \langle\theta\rangle^2\right) - 2\left\langle\sigma^2\right\rangle. \tag{5}$$

3 LRS spacetime

Locally rotationally symmetric (LRS) spacetimes are defined by the following characterization [4]: In an open neighborhood of each point p, there is a nondiscrete subgroup of the Lorentz group which leaves the Riemann tensor and its covariant derivatives to the third order invariant. There is therefore in LRS spacetimes a preferred direction e^μ (the axis of symmetry) in every point.

We will use the covariant 3+1 splitting of a spacetime with the timelike vector u^μ normalized by the condition $u_\rho u^\rho = -1$ and the projection tensor $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. Because of the property of the LRS spacetime, all covariantly defined spacelike vectors orthogonal to u^μ must be proportional to e^μ [5].

$$\dot{u}^\mu = \dot{u}e^\mu, \quad \omega^\mu = \omega e^\mu, \quad h^\sigma_\mu \nabla_\sigma \rho = \rho' e_\mu, \quad h^\sigma_\mu \nabla_\sigma p = p' e_\mu, \; h^\sigma_\mu \nabla_\sigma \theta = \theta' e_\mu$$

Dot here denotes the covariant derivative along the flow vector u^μ and the prime denotes covariant derivative along the vector e^μ. With the help of the tensor $e_{\mu\nu} = \frac{1}{3}(3e_\mu e_\nu - h_{\mu\nu})$, we have the relations for the shear tensor and the electric and magnetic parts of the Weyl tensor

$$\sigma_{\mu\nu} = \frac{2}{\sqrt{3}}\sigma e_{\mu\nu}, \quad E_{\mu\nu} = \frac{2}{\sqrt{3}}E e_{\mu\nu}, \quad H_{\mu\nu} = \frac{2}{\sqrt{3}}H e_{\mu\nu}. \tag{6}$$

We will now define the magnitude of the spatial rotation k and the magnitude of the spatial divergence a,

$$k \equiv \left|\eta^{\alpha\beta\gamma\delta}\left(\nabla_\beta e_\gamma\right)u_\delta\right|, \qquad a \equiv h^\alpha_\beta\left(\nabla_\alpha e^\beta\right). \tag{7}$$

4 Averaging LRS spacetime

For simplicity we will restrict to the class II LRS spacetime with the condition $p = 0$ (dust models) which includes LTB spacetimes and their generalizations to spacelike 2-surfaces with negative or zero curvature scalar. Given a preferred spacelike direction, all the equations describing LRS metric are scalar. It means we can perform averaging (which is covariantly defined for scalars). In order to obtain averaged equations we need to derive the commutation relations for the prime (and the time) derivative (with respect to the preferred direction) and averaging,

$$\langle A\rangle'_{\mathscr{D}} = \mathbf{e}\left(\frac{1}{V_{\mathscr{D}}}\int_{\mathscr{D}} d^3x\sqrt{\det g_{ij}}A\right) = -\langle\xi\rangle_{\mathscr{D}}\langle A\rangle_{\mathscr{D}} + \langle A\xi\rangle_{\mathscr{D}} + \langle A'\rangle_{\mathscr{D}}, \tag{8}$$

where ξ is defined by the relation $\left(\sqrt{\det g_{ij}}\right)' = \sqrt{\det g_{ij}}\xi$. Similarly we get the commutation rule for the time derivative. Averaged equations for the dust class II LRS spacetime read

$$\begin{aligned}
\langle\theta\rangle^{\cdot} &= -\frac{1}{3}\langle\theta\rangle^2 - 4\pi\langle\rho\rangle + \underline{\frac{2}{3}\left(\langle\theta^2\rangle - \langle\theta\rangle^2\right) - 2\langle\sigma^2\rangle}\\
\langle\sigma\rangle^{\cdot} &= -\frac{1}{\sqrt{3}}\langle\sigma\rangle^2 - \frac{2}{3}\langle\theta\rangle\langle\sigma\rangle - \langle E\rangle + \underline{\frac{1}{\sqrt{3}}\left(\langle\sigma\rangle^2 - \langle\sigma^2\rangle\right) + \frac{1}{3}\left(\langle\theta\sigma\rangle - \langle\theta\rangle\langle\sigma\rangle\right)}\\
\langle E\rangle^{\cdot} &= -4\pi\langle\rho\rangle\langle\sigma\rangle + \sqrt{3}\langle E\rangle\langle\sigma\rangle - \langle\theta\rangle\langle E\rangle - \underline{4\pi\left(\langle\rho\sigma\rangle - \langle\rho\rangle\langle\sigma\rangle\right)}\\
&\quad + \underline{\sqrt{3}\left(\langle E\sigma\rangle - \langle E\rangle\langle\sigma\rangle\right)}\\
\langle\rho\rangle^{\cdot} &= -\langle\rho\rangle\langle\theta\rangle\\
\langle a\rangle^{\cdot} &= -\frac{1}{3}\langle a\rangle\langle\theta\rangle + \frac{1}{\sqrt{3}}\langle a\rangle\langle\sigma\rangle + \underline{\frac{2}{3}\left(\langle a\theta\rangle - \langle a\rangle\langle\theta\rangle\right) + \frac{1}{\sqrt{3}}\left(\langle a\sigma\rangle - \langle a\rangle\langle\sigma\rangle\right)}\\
\langle\sigma\rangle' &= \frac{1}{\sqrt{3}}\langle\theta\rangle' - \frac{2}{3}\langle a\rangle\langle\sigma\rangle + \underline{\langle\sigma\xi\rangle - \langle\xi\rangle\langle\sigma\rangle - \frac{1}{\sqrt{3}}\left(\langle\xi\theta\rangle - \langle\xi\rangle\langle\theta\rangle\right)}\\
&\quad \underline{-\frac{3}{2}\left(\langle a\sigma\rangle - \langle a\rangle\langle\sigma\rangle\right)}
\end{aligned}$$

$$\begin{aligned}
\langle E\rangle' =& -\frac{2}{3}\langle a\rangle\langle E\rangle+\frac{4\pi}{\sqrt{3}}\langle\rho\rangle'-\underline{\frac{2}{3}\left(\langle aE\rangle-\langle a\rangle\langle E\rangle\right)}\\
&\underline{+\langle\xi E\rangle-\langle\xi\rangle\langle E\rangle-\frac{4\pi}{\sqrt{3}}\left(\langle\xi\rho\rangle-\langle\xi\rangle\langle\rho\rangle\right)}\\
\langle a\rangle' =& \frac{2}{9}\langle\theta\rangle^2+\frac{2}{3\sqrt{3}}\langle\theta\rangle\langle\sigma\rangle-\frac{4}{3}\langle\sigma\rangle^2-\frac{2}{\sqrt{3}}\langle E\rangle-\frac{1}{2}\langle a\rangle^2-\frac{16\pi}{3}\langle\rho\rangle\\
&\underline{+\langle a\xi\rangle-\langle a\rangle\langle\xi\rangle+\frac{2}{9}\left(\langle\theta^2\rangle-\langle\theta\rangle^2\right)+\frac{2}{3\sqrt{3}}\left(\langle\theta\sigma\rangle-\langle\theta\rangle\langle\sigma\rangle\right)}\\
&\underline{-\frac{4}{3}\left(\langle\sigma^2\rangle-\langle\sigma\rangle^2\right)-\frac{1}{2}\left(\langle a^2\rangle-\langle a\rangle^2\right)}
\end{aligned}\tag{9}$$

The underlined part of the equations denotes the additional terms created by averaging. We can recognize the familiar Buchert equation with the kinematical backreaction term and the mass conservation equation.

5 Conclusion

We have shown how to generalize the Buchert equations for the LRS spacetimes. Averaged Einstein equations consist of evolution equations and constraints that are preserved in time.

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