Einstein's "Prague Field Equation" of 1912 – Another Perspective

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Abstract I reconsider Einstein's 1912 "Prague-Theory" of static gravity based on a scalar field obeying a non-linear field equation. I point out that this equation follows from the self-consistent implementation of the principle that *all* forms of energy are the source of the gravitational field according to $E = mc^2$. This makes it an interesting toy-model for the "flat-space approach" to General Relativity (GR), as pioneered by Kraichnan and later Feynman. Solutions modelling stars show features familiar from GR, e.g., Buchdahl-like inequalities. The relation to full GR is also discussed. This lends this toy theory also some pedagogical significance.

1 Introduction

Ever since he wrote his large 1907 review of Special Relativity [1] for the *Jahrbuch der Radioaktivität und Elektronik*, Einstein reflected on how to extend the principle of relativity to non-inertial motions. His key insight was that such an extension is indeed possible, provided gravitational fields are included in the description. In fact, the last chapter (V) of [1], which comprises four (17-20) out of twenty sections, is devoted to this intimate relation between acceleration and gravitation. The heuristic principle Einstein used was his "Äquivalenzhypothese" (hypothesis of equivalence) or "Äquivalenzprinzip" (principle of equivalence)[1], which says this: Changing the description of a system from an inertial to a non-inertial reference frame is equivalent to not changing the frame at all but adding a *special* gravitational field. This principle is *heuristic* in the sense that it allows to deduce the extension of physical

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[1] In his Prague papers Einstein gradually changed from the first to the second expression.

laws, the forms of which are assumed to be known in the absence of gravitational fields, to the presence of at least those special gravitational fields that can be "created" by mere changes of reference frames. The idea behind this was, of course, to postulate that the general features found in this fashion remain valid in *all* gravitational fields. In the 1907 review Einstein used this strategy to find out about the influence gravitational fields have on clocks and general electromagnetic processes. What he did not attempt back in 1907 was to find an appropriate law for the gravitational field that could replace the Poisson equation of Newtonian gravity. This he first attempted in his two "Prague papers" from 1912 [2][3] for static fields. The purpose of my contribution here is to point out that the field equation Einstein arrived at in the second of these papers is not merely of historical interest.

After 1907 Einstein turned away from gravity research for a while, which he resumed in 1911 with a paper [4], also from Prague, in which he used the "Äquivalenzhypothese" to deduce the equality between gravitational and inertial mass, the gravitational redshift, and the deflection of light by the gravitational field of massive bodies. As is well known, the latter resulted in half the amount that was later correctly predicted by GR.

In the next gravity paper [2], the first in 1912, entitled "*Lichtgeschwindigkeit und Statik des Gravitationsfeldes*", Einstein pushed further the consequences of his heuristics and began his search for a sufficiently simple differential equation for static gravitational fields. The strategy was to, first, guess the equation from the form of the special fields "created" by non inertial reference frames and, second, generalise it to those gravitational fields generated by real matter. Note that the gravitational acceleration was to be assumed to be a gradient field (curl free) so that the sought-after field equation was for a scalar field, the gravitational potential.

The essential idea in the first 1912 paper is to identify the gravitational potential with c, the local velocity of light.[2] Einstein's heuristics indicated clearly that Special Relativity had to be abandoned, in contrast to the attempts by Max Abraham (1875-1922), who published a rival theory [5][6] that was superficially based on Poincaré invariant equations (but violated Special Relativity in abandoning the condition that the four-velocities of particles had constant Minkowski square). In passing I remark that Einstein's reply [7] to Abraham, which is his last paper from Prague before his return to Zürich, contains in addition to his anticipation of the essential physical hypotheses on which a future theory of gravity could be based (here I refer to Jiří Bičák's contribution to this volume), also a concise and very illuminating account of the physical meaning and limitation of the special principle of relativity, the essence of which was totally missed by Abraham.

Back to Einstein's first 1912 paper, the equation he came up with was

$$\Delta c = kc\rho\,, \tag{1}$$

[2] Since here we will be more concerned with the mathematical form and not so much the actual derivation by Einstein, we will ignore the obvious objection that c has the wrong physical dimension, namely that of a velocity, whereas the proper gravitational potential should have the dimension of a velocity-squared.

where k is the "universal gravitational constant" and ρ is the mass density. The mathematical difference between (1) and the Poisson equation in Newtonian gravity is that (1) is homogeneous (even linear) in the potential. This means that the source strength of a mass density is weighted by the gravitational potential at its location. This implies a kind of "red-shift" for the active gravitational mass which in turn results in the existence of geometric upper bounds for the latter, as we will discuss in detail below. Homogeneity was Einstein's central requirement, which he justified from the interpretation of the gravitational potential as the local velocity of light, which is only determined up to constant rescalings induced from rescalings of the timescale.

Already in a footnote referring to equation (1) Einstein points out that it cannot be quite correct, as he is to explain in detail in a follow-up paper [3]. This second paper of 1912 is the one I actually wish to focus on in my contribution here. It appeared in the same issue of the *Annalen der Physik* as the previous one, under the title "*Zur Theorie des statischen Gravitationsfeldes*" (on the theory of the static gravitational field). In it Einstein once more investigates how the gravitational field influences electromagnetic and thermodynamic processes according to what he now continues to call the "*Aquivalenzprinzip*", and derives from it the equality of inertial and gravitational mass.[3]

After that he returns to the equation for the static gravitational field and considers the gravitational force-density $\mathbf{f}$, acting on ponderable matter of mass density ρ, which is given by (Einstein writes σ instead of our ρ)

$$\mathbf{f} = -\rho \nabla c \,. \tag{2}$$

Einstein observes that the space integral of $\mathbf{f}$ does not necessarily vanish on account of (1), in violation of the principle that *actio* equals *reactio*. Terrible consequences, like self-acceleration, have to be envisaged.[4] He then comes up with the following non-linear but still homogeneous modification of (1):

[3] Einstein considers radiation enclosed in a container whose walls are "massless" (meaning vanishing rest-mass) but can support stresses, so as to be able to counteract radiation pressure. Einstein keeps repeating that equality of both mass types can only be proven if the gravitational field does not act on the stressed walls. That remark is hard to understand in view of the fact that unbalanced stresses add to inertia, as he well knew from his own earlier investigations [8]. However, as explained by Max Laue a year earlier [9], the gravitational action on the stressed walls is just cancelled by that on the stresses of the electromagnetic field, for both systems together form a "complete static system", as Laue calls it. A year later, in the 1913 "Entwurf" paper with Marcel Grossmann [10], Einstein again used a similar Gedankenexperiment with a massless box containing radiation immersed in a gravitational field, by means of which he allegedly shows that any Poincaré invariant scalar theory of gravity must violate energy conservation. A modern reader must ask how this can possibly be, in view of Noether's theorem applied to time-translation invariance. A detailed analysis [11] shows that this energy contains indeed the expected contribution from the tension of the walls, which may not be neglected.

[4] "*Anderenfalls würde sich die Gesamtheit der in dem betrachteten Raume befindlichen Massen, die wir auf einem starren, masselosen Gerüste uns befestigt denken wollen, sich in Bewegung zu setzen streben.*" ([3], p. 452)

$$\Delta c = k\left\{c\rho + \frac{1}{2k}\frac{\nabla c\cdot\nabla c}{c}\right\}. \tag{3}$$

In the rest of this paper we will show how to arrive at this equation from a different direction and discuss some of its interesting properties as well as its relation to the description of static gravitational fields in GR.

2 A self-consistent modification of Newtonian Gravity

The following considerations are based on [12]. We start from ordinary Newtonian gravity, where the gravitational field is described by a scalar function ϕ whose physical dimension is that of a velocity-squared. It obeys

$$\Delta\varphi = 4\pi G\rho\,. \tag{4}$$

The force per unit volume that the gravitational field exerts upon a distribution of matter with density ρ is

$$\mathbf{f} = -\rho\nabla\varphi\,. \tag{5}$$

This we apply to the force that the gravitational field exerts upon its own source during a real-time process of redistribution. This we envisage as actively transporting each mass element along the flow line of a vector field ξ. To first order, the change $\delta\rho$ that ρ suffers in time δt is given by

$$\delta\rho = \frac{-L_{\delta\xi}\left(\rho d^3x\right)}{d^3x} = -\nabla\cdot(\delta\xi\,\rho)\,, \tag{6}$$

where $\delta\xi = \delta t\,\xi$ and $L_{\delta\xi}$ is the Lie derivative with respect to $\delta\xi$. We assume the support $\mathrm{supp}(\rho) =: B \subset \mathbb{R}^3$ to be compact. In general, this redistribution costs energy. The work we have to invest for redistribution is, to first order, just given by

$$\delta A = -\int_{\mathbb{R}^3}\delta\xi\cdot\mathbf{f} = -\int_B \varphi\nabla\cdot(\delta\xi\,\rho) = \int_B \varphi\,\delta\rho\,, \tag{7}$$

where we used (6) in the last step and where we did not write out the Lebesgue measure d^3x to which all integrals refer. Note that in order to obtain (7) we did not make use of the field equation. Equation (7) is generally valid whenever the force-density relates to the potential and the mass density as in (5).

Now we make use of the field equation (4). We assume the redistribution-process to be adiabatic, that is, we assume the instantaneous validity of the field equation at each point in time throughout the process. This implies

$$\Delta\delta\varphi = 4\pi G\,\delta\rho\,. \tag{8}$$

Hence, using (7), the work invested in the process of redistribution is (to first order)

$$\delta A = \int_B \varphi\,\delta\rho = \delta\left\{-\frac{1}{8\pi G}\int_{\mathbb{R}^3}(\nabla\varphi)^2\right\}. \tag{9}$$

If the infinitely dispersed state of matter is assigned the energy-value zero, then the expression in curly brackets is the total work invested in bringing the infinitely dispersed state to that described by the distribution ρ. This work must be stored somewhere as energy. Like in electro-statics and -dynamics, we take a further logical step and assume this energy to be spatially distributed in the field according to the integrand. This leads to the following expression for the energy density of the static gravitational field

$$\varepsilon = -\frac{1}{8\pi G}(\nabla\varphi)^2. \tag{10}$$

All this is familiar from Newtonian gravity. But now we go beyond Newtonian gravity and require the validity of the following

Principle. *All forms of energy, including that of the gravitational field itself, shall gravitate according to* $E = mc^2$.

This principle implies that if we invest an amount of work δA in a system its (active) gravitational mass will increase by $\delta A/c^2$.

Now, the (active) gravitational mass M_g is defined by the flux of the gravitational field to spatial infinity (i.e. through spatial spheres as their radii tend to infinity):

$$M_g = \frac{1}{4\pi G}\int_{S^2_\infty}\mathbf{n}\cdot\nabla\varphi = \frac{1}{4\pi G}\int_{\mathbb{R}^3}\Delta\varphi. \tag{11}$$

Hence, making use of the generally valid equation (7), the principle that $\delta A = M_g c^2$ takes the form

$$\int_B \varphi\,\delta\rho = \frac{c^2}{4\pi G}\int_{\mathbb{R}^3}\Delta\delta\varphi. \tag{12}$$

This functional equation relates φ and ρ, over and above the restriction imposed on their relation by the field equation. However, the latter may - and generally will - be inconsistent with this additional equation. For example, the Newtonian field equation (4) is easily seen to manifestly violate (12), for the right-hand side then becomes just the integral over $c^2\delta\rho$, which always vanishes on account of (6) (or the obvious remark that the redistribution clearly does not change the total mass), whereas the left hand side will generally be non-zero. The task must therefore be to find field equation(s) consistent with (12). Our main result in that direction is that the unique generalisation of (4) which satisfies (12) is just (3), i.e. the field equation from Einstein's second 1912 paper.

Let us see how this comes about. A first guess for a consistent modification of (4) is to simply add ε/c^2 to the source ρ:

$$\Delta\varphi = 4\pi G\left(\rho - \frac{1}{8\pi Gc^2}(\nabla\varphi)^2\right). \tag{13}$$

But this cannot be the final answer because this change of the field equation also brings about a change in the expression for the self-energy of the gravitational field. That is, the term in the bracket on the right-hand side is not the total energy according to *this* equation, but according to the original equation (4). In other words: equation (13) still lacks *self*-consistency. This can be corrected for by iterating this procedure, i.e., determining the field's energy density according to (13) and correcting the right-hand side of (13) accordingly. Again we have changed the equation, and this goes on ad infinitum. But the procedure converges to a unique field equation, similarly to the convergence of the "Noether-procedure"[5] that leads from the Poincaré invariant Pauli-Fierz theory of spin-2 mass-0 fields in flat Minkowski space to GR [13][14][15].

In our toy model the convergence of this procedure is not difficult to see. We start from the definition (11) and calculate its variation δM_g assuming the validity of (13). From what we said above we know already that this is not yet going to satisfy (12). But we will see that from this calculation we can read off the right redefinitions.

We start by varying (11):

$$\delta M_g = \frac{1}{4\pi G}\int \Delta\delta\varphi\,. \tag{14}$$

We replace $\Delta\delta\varphi$ with the variation of the right-hand side of (13). Partial integration of the non-linear part gives us a surface term whose integrand is $\propto \varphi\nabla\delta\varphi = O(r^{-3})$ and hence vanishes. The remaining equation is

$$\delta M_g = \int_B \delta\rho + \frac{1}{4\pi G}\int_{\mathbb{R}^3}\left(\frac{\varphi}{c^2}\right)\Delta\delta\varphi\,. \tag{15}$$

Playing the same trick (of replacing $\Delta\delta\varphi$ with the variation of the right-hand side of (13) and partial integration, so as to collect all derivatives on $\delta\varphi$) again and again, we arrive after N steps at

$$\delta M_g = \int_B \sum_{n=0}^{N-1}\frac{1}{n!}\left(\frac{\varphi}{c^2}\right)^n\delta\rho + \frac{1}{N!c^{2N}}\frac{1}{4\pi G}\int_{\mathbb{R}^3}\varphi^N\delta(\Delta\varphi)\,. \tag{16}$$

As φ is bounded for a regular matter distribution, and the spatial integral over $\delta\Delta\varphi$ is just $4\pi G\delta M_g$, the last term tends to zero for $N\to\infty$. Hence

$$\delta M_g = \int_B \delta\rho\,\exp(\varphi/c^2)\,. \tag{17}$$

This *is* of the desired form (12) required by the principle, provided we redefine the gravitational potential to be Φ rather than φ, where

$$\Phi := c^2\exp(\varphi/c^2)\,. \tag{18}$$

[5] Pioneered by Robert Kraichnan in his 1947 MIT Bachelor thesis "Quantum Theory of the Linear Gravitational Field".

Saying that Φ rather than φ is the right gravitational potential means that the force density is not given by (5), but rather by

$$\mathbf{f} = -\rho\nabla\Phi\,. \tag{19}$$

As we have made use of equation (13) in order to derive (17), we must make sure to keep *that* equation, just re-expressed in terms of Φ. This leads to

$$\Delta\Phi = \frac{4\pi G}{c^2}\left[\rho\Phi + \frac{c^2}{8\pi G}\frac{(\nabla\Phi)^2}{\Phi}\right] \tag{20}$$

which is precisely Einstein's improved "Prague equation" (3) with $k = 4\pi G/c^2$. Note from (18) that the asymptotic condition $\varphi(r\to\infty)\to 0$ translates to $\Phi(r\to\infty)\to c^2$. Note also that for $r\to\infty$ the $1/r^2$-parts of $\nabla\varphi$ and $\nabla\Phi$ coincide, so that in the expressions (11) for M_g we may just replace φ with Φ:

$$M_g = \frac{1}{4\pi G}\int_{S^2_\infty}\mathbf{n}\cdot\nabla\Phi = \frac{1}{4\pi G}\int_{\mathbb{R}^3}\Delta\Phi\,. \tag{21}$$

The principle now takes the form (12) with φ replaced by Φ. It is straightforward to show by direct calculation that (12) is indeed a consequence of (20), as it must be. It also follows from (20) that the force density (19) is the divergence of a symmetric tensor:

$$f_a = -\nabla^b t_{ab}\,, \tag{22a}$$

where

$$t_{ab} = \frac{1}{4\pi Gc^2}\left\{\frac{1}{\Phi}\left[\nabla_a\Phi\nabla_b\Phi - \tfrac{1}{2}\delta_{ab}(\nabla\Phi)^2\right]\right\}. \tag{22b}$$

This implies the validity of the principle that actio equals reactio that Einstein demanded. *This* was Einstein's rationale for letting (3) replace (1).

Finally we mention that (20) may be linearised if written in terms of the square-root of Φ:

$$\Psi := \sqrt{\frac{\Phi}{c^2}}\,. \tag{23}$$

One gets

$$\Delta\Psi = \frac{2\pi G}{c^2}\rho\Psi\,. \tag{24}$$

This helps in finding explicit solutions to (20). Note that Ψ is dimensionless.

3 Spherically symmetric solutions

In this section we discuss some properties of spherically symmetric solutions to (24) for spherically symmetric mass distributions ρ of compact support. In the following we will simply refer to the object described by such a mass distribution as "star".

In terms of $\chi(r) := r\Psi(r)$ equation (24) is equivalent to

$$\chi'' = \frac{2\pi G}{c^2}\,\rho\,\chi\,. \tag{25}$$

The support of ρ is a closed ball of radius R, called the star's radius. For $r < R$ we shall assume $\rho(r) \geq 0$ (weak energy condition). We seek solutions which correspond to everywhere positive and regular Ψ and hence everywhere positive and regular Φ. In particular $\Phi(r=0)$ and $\Psi(r=0)$ must be finite. For $r > R$ equation (25) implies $\chi'' = 0$, the solution to which is

$$\chi_+(r) = r\Psi_+(r) = r - R_g\,, \quad \text{for } r > R\,, \tag{26}$$

where R_g denotes the gravitational radius

$$R_g := \frac{GM_g}{2c^2}\,. \tag{27}$$

R_g comes in because of (21), which fixes one of the two integration constants, the other being fixed by $\Psi(\infty) = 1$.

Let χ_- denote the solution in the interior of the star. Continuity and differentiability at $r = R$ gives $\chi_-(R) = R - R_g$ and $\chi'_-(R) = 1$. We observe that $\chi_-(R) \geq 0$. For suppose $\chi_-(R) < 0$, then (25) and the weak energy condition imply $\chi''(R) \leq 0$. But this implies that for $r \in [0,R]$ the curve $r \mapsto \chi_-(r)$ lies below the straight line $r \mapsto r - R_g$ and assumes a value less than $-R_g$ at $r = 0$, in contradiction to the finiteness of $\Psi(r=0)$ which implies $\chi_-(r=0) = 0$. Hence we have

Theorem 1. *The gravitational radius of a spherically symmetric star is universally bound by its (geometric) radius, $R_g \leq R$. Equivalently expressed in terms of M_g we may say that the gravitational mass is universally bound above by*

$$M_g < \frac{2c^2 R}{G}\,. \tag{28}$$

This may be seen in analogy to Buchdahl's inequality in GR [16], which, using the isotropic (rather than Schwarzschild) radial coordinate, would differ from (28) only by an additional factor of $8/9$ on the right-hand side. The Buchdahl bound is optimal, being saturated by the interior Schwarzschild solution for a homogeneous star.

So let us here, too, specialise to a homogeneous star,

$$\rho(r) = \begin{cases} \frac{3M_b}{4\pi R^3} & \text{for } r \leq R \\ 0 & \text{for } r > R\,, \end{cases} \tag{29}$$

where M_b is called the bare mass (integral over ρ). It is convenient to introduce the radii corresponding to bare and gravitational masses, as well as their ratio to the star's radius R:

$$R_b := \frac{GM_b}{2c^2}\,, \quad x := \frac{R_b}{R}\,, \tag{30a}$$

$$R_g := \frac{GM_g}{2c^2}\,, \quad y := \frac{R_g}{R}\,. \tag{30b}$$

We also introduce the inverse length

$$\omega := \frac{1}{R}\cdot\sqrt{\frac{3R_b}{R}}\,, \tag{31}$$

so that (25) just reads $\chi'' = \omega^2\chi$. From this the interior solution is easily obtained. If written in terms of Ψ it reads

$$\Psi_-(r) = \frac{1}{\cosh(\omega R)}\frac{\sinh(\omega r)}{\omega r}\,, \quad \text{for } r < R\,. \tag{32}$$

As a result of the matching to the exterior solution given in (26), R_g is determined by R and ω, i.e. R and R_b. In terms of x and y this relation takes the simple form

$$y = 1 - \frac{\tanh\left(\sqrt{3x}\right)}{\sqrt{3x}}\,, \tag{33}$$

which convex-monotonically maps $[0,\infty)$ onto $[0,1)$. The fact that $y < 1$ for all x is just the statement of the Theorem applied to the homogeneous case.

If $x = R_b/R \ll 1$ we have $y = x - \frac{6}{5}x^2 + \cdots$, which for $E_{\text{total}} := M_g c^2$ reads

$$E_{\text{total}} = M_b c^2\left(1 - \tfrac{3}{5}x + O(x^2)\right)\,. \tag{34}$$

We note that $-3M_b c^2 x/5 = -\frac{3}{5}GM_b^2/R$ is just the Newtonian binding energy of a homogeneous star. In view of our Principle it makes good sense that to first order just this amount is subtracted from the bare mass in order to obtain the active gravitational mass. In Newtonian gravity this negative amount is just identified with the field's self-energy, but here the interpretation is different: The two terms that act as source for the gravitational field in (20) are the matter part, which is proportional to ρ but diminished by Φ, and the field's own part, which is proportional to $(\nabla\Phi)^2/\Phi$ and positive definite! Their contributions are, respectively,

$$E_{\text{matter}} = \int_B \rho\Phi = M_b c^2\left(1 - \tfrac{6}{5}x + O(x^2)\right)\,, \tag{35}$$

$$E_{\text{field}} = \frac{c^2}{8\pi G}\int_{\mathbb{R}^3}\frac{(\nabla\Phi)^2}{\Phi} = M_b c^2\left(\tfrac{3}{5}x + O(x^2)\right)\,. \tag{36}$$

Hence even though the total energy is decreased due to binding, the gravitational field's self energy *increases* by the same amount. Twice that amount is gained from the fact that the matter-energy is "red-shifted" by being multiplied with Φ, so energy is conserved (of course).

Two more consequences, which are related, are noteworthy:

- Unlike in Newtonian theory, objects with non-zero gravitational mass cannot be modelled by point sources. In the spherically symmetric case this is an immediate consequence of (28), which implies $M_g \to 0$ for $R \to 0$. Hence there are no δ-like masses.
- Unlike in Newtonian gravity, unlimited compression of matter does not lead to unlimited energy release. Consider a sequence of homogeneous (just for simplicity) stars of fixed bare mass M_b and variable radius R, then the gravitational mass M_g as function of $x = R_b/R$ is given by

$$M_g(x) = M_b \cdot \left\{ \frac{1}{x} \cdot \left(1 - \frac{\tanh\left(\sqrt{3x}\right)}{\sqrt{3x}} \right) \right\} . \tag{37}$$

  The function in curly brackets[6] is a strictly monotonically decreasing function $[0,\infty] \mapsto [1,0]$. This shows that for infinitely dispersed matter, where $R \to \infty$ and hence $x \to 0$, we have $M_g(x=0) = M_b$, as expected, and that for infinite compression $M_g(x \to \infty) = 0$. As the gained energy at stage x is $(M_b - M_g(x))c^2$, we can at most gain $M_b c^2$.

4 Relation to General Relativity

Finally I wish to briefly comment on the relation of equation (3) or (20) to GR. Since Einstein's 1912 theory was only meant to be valid for static situations, I will restrict attention to static spacetimes (M,g). Hence I assume the existence of a timelike and hypersurface orthogonal Killing field K. My signature convention shall be "mostly plus", i.e. $(-,+,+,+)$.

We choose adapted coordinates (t,x^a), $a = 1,2,3$, where the level sets of t are the integral manifolds of the foliation defined by K and $K = \partial/\partial(ct)$. We can then write the metric in a form in which the coefficients do not depend on t (called "time") ,

$$g = -\Psi^2(x)\, c^2\, dt \otimes dt + \hat{g}_{ab}(x)\, dx^a \otimes dx^b \, . \tag{38}$$

Clearly $c^2\Psi^2 = -g(K,K)$. From now on, all symbols with hats refer to the spatial geometry, like the spatial metric $\hat{g}$.

The t-component of the geodesic equation is equivalent to $\Psi^2 \dot{t} = \text{const}$, where an overdot refers to the derivative with respect to an affine parameter. This equation allows us to eliminate the affine parameter in favour of t in the spatial components of the geodesic equation. If we set[7]

$$\Psi = \sqrt{\frac{2\Phi}{c^2}} \tag{39}$$

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[6] Its Taylor expansion at $x = 0$ is $1 - 6x/5 + 51x^2/35 + \cdots$.

[7] This differs by a factor of 2 from (23) which we need and to which we return below.

they read

$$\frac{d^2x^a}{dt^2} + \hat{\Gamma}^a_{bc}\frac{dx^b}{dt}\frac{dx^c}{dt} = -\Phi_{,b}\hat{g}^{ab} + \Phi_{,b}\left[\frac{1}{\Phi}\frac{dx^a}{dt}\frac{dx^b}{dt}\right], \tag{40}$$

where the $\hat{\Gamma}^a_{bc}$ are the Christoffel coefficients for $\hat{g}$, and $\Phi_{,a} = \partial_a\Phi$. This should be compared with (19) together with Newton's second law, which give $d^2\mathbf{x}/dt^2 = -\nabla\Phi$. As we did not attempt to include special relativistic effects in connection with high velocities, we should consistently neglect terms v^2/c^2 in (40). This results in dropping the rightmost term. The rest has the pseudo-Newtonian form in arbitrary (not just inertial) spatial coordinates. A non-zero spatial curvature would, of course, be a new feature not yet considered.

The curvature and Ricci tensors for the metric (38) are readily computed, most easily by using Cartan's structure equations:

$$\mathrm{Ric}(n,n) = \Psi^{-1}\hat{\Delta}\Psi, \quad R_{ab} = \hat{R}_{ab} - \Psi^{-1}\hat{\nabla}_a\hat{\nabla}_b\Psi. \tag{41}$$

Here $n = \Psi^{-1}\partial/c\partial t$ is the unit timelike vector characterising the static reference frame, $\hat{\nabla}$ is the Levi-Civita covariant derivative with respect to $\hat{g}$, and $\hat{\Delta}$ is the corresponding Laplacian.

Using this in Einstein's equations

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T^\lambda_\lambda\right) \tag{42}$$

for pressureless (we neglect the pressure since it enters multiplied with c^{-2}) dust at rest and of mass-density ρ in the static frame, i.e.

$$T_{\mu\nu} = \rho c^2 n_\mu n_\nu, \tag{43}$$

we get

$$\hat{\Delta}\Psi = \frac{4\pi G}{c^2}\rho\Psi \qquad \text{time component}, \tag{44a}$$

$$\hat{\nabla}_a\hat{\nabla}_b\Psi = \hat{R}_{ab}\Psi \qquad \text{space components}. \tag{44b}$$

We note that, apart from the space curvature, (44a) is almost—but not quite—identical to (24). They differ by a factor of 2! Rewriting (44a) in terms of Φ according to (39), we get

$$\hat{\Delta}\Phi = \frac{8\pi G}{c^2}\left[\rho\Phi + \frac{c^2}{16\pi G}\frac{\hat{g}^{ab}\hat{\nabla}_a\Phi\hat{\nabla}_b\Phi}{\Phi}\right]. \tag{45}$$

This differs from (20) by the same factor of 2 (i.e., $G \to 2G$). Note that we cannot simply remove this factor by rescaling Ψ and Φ, as the equations are homogeneous in these fields. Note also that the overall scale of Φ is fixed by (40): It is the gradient of Φ, and not a multiple thereof, which gives the acceleration. But then there is another factor of 2 in difference to our earlier discussion: If the metric (38) is to approach the Minkowski metric far away from the source, then Ψ should tend to one and hence Φ should asymptotically approach $c^2/2$ according to (39). In (20), however, Φ should asymptotically approach c^2, i.e. twice that value. This additional factor of 2 ensures that both theories have the same Newtonian limit. Indeed, if we expand the gravitational potential Φ of an isolated object in a power series in G, this implies that the linear terms of both theories coincide. However, the quadratic terms in GR are twice as large as in our previous theory based on (19) and (20). This is not quite unexpected if we take into account that in GR we also have the space curvature that will modify the fields and geodesics in post Newtonian approximations. We note that the spatial Einstein equations (44b) prevent space from being flat. For example, taking their trace and using (44a) shows that the scalar curvature of space is, in fact, proportional to the mass density.

Finally we show that the total gravitational mass in GR is just given by the same formula (21), where Φ is now that used here in the GR context. To see this we recall that for spatially asymptotically flat spacetimes the overall mass (measured at spatial infinity) is given by the ADM-mass. Moreover, for spatially asymptotically flat spacetimes which are stationary and satisfy Einstein's equations with sources of spatially compact support, the ADM mass is given by the Komar integral (this is, e.g., proven in Theorem 4.13 of [17]). Hence we have

$$M_{\mathrm{ADM}} = \frac{c^2}{8\pi G}\int_{S^2_\infty} \star dK^\flat \,. \tag{46}$$

Here $K = \partial/\partial(ct)$, and $K^\flat := g(K,\cdot) = -\Psi^2 c dt$ is the corresponding 1-form. The star, $\star$, denotes the Hodge-duality map. Using (39) and asymptotic flatness it is now straightforward to show that the right hand side of (46) can indeed be written in the form of the middle term in (21). This term only depends on Φ at infinity, i.e. on the Newtonian limit, and hence it gives a value independent of the factor-2 discrepancy discussed above. In that sense the active gravitational mass M_g defined earlier corresponds to M_{ADM} in the GR context.

This ends our discussion of Einstein's 1912 scalar field equation, which is thus seen to contain many interesting features we know from GR, albeit in a pseudo Newtonian setting.

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