Canonical Superenergy Tensors in General Relativity: a Reappraisal

Janusz Garecki

Abstract We discuss the role of the canonical superenergy tensors.

1 Introduction

In the framework of general relativity (**GR**), as a consequence of the Einstein Equivalence Principle (**EEP**), the gravitational field *has non-tensorial strengths* $\Gamma^i_{kl} = \{^{\,i}_{kl}\}$ and *admits no energy-momentum tensor*. One can only attribute to this field *gravitational energy-momentum pseudotensors*. The leading object of such a kind is the *canonical gravitational energy-momentum pseudotensor* ${}_E t_i^{\ k}$ proposed already in past by Einstein. This pseudotensor is a part of the *canonical energy-momentum complex* ${}_E K_i^{\ k}$ in **GR**.

The canonical complex ${}_E K_i^{\ k}$ can easily be obtained by rewriting Einstein equations to the superpotential form

$$ {}_E K_i^{\ k} := \sqrt{|g|}\big(T_i^{\ k} + {}_E t_i^{\ k}\big) = {}_F U_i^{\ [kl]}{}_{,l}\,, \tag{1}$$

where $T^{ik} = T^{ki}$ is the symmetric energy-momentum tensor for matter, $g = det[g_{ik}]$, and

$$\begin{aligned} {}_E t_i^{\ k} = \frac{c^4}{16\pi G}\big\{&\delta^k_i g^{ms}\big(\Gamma^l_{mr}\Gamma^r_{sl} - \Gamma^r_{ms}\Gamma^l_{rl}\big) \\ &+ g^{ms}{}_{,i}\big[\Gamma^k_{ms} - \frac{1}{2}\big(\Gamma^k_{tp}g^{tp} - \Gamma^l_{tl}g^{kt}\big)g_{ms} \\ &- \frac{1}{2}\big(\delta^k_s\Gamma^l_{ml} + \delta^k_m\Gamma^l_{sl}\big)\big]\big\}\,; \end{aligned}$$

Janusz Garecki
Institute of Mathematics University of Szczecin and Cosmology Group University of Szczecin, Wielkopolska 15, 70-451 Szczecin, Poland e-mail: garecki@wmf.univ.szczecin.pl

$$ {}_{F}U_i^{\ [kl]} = \frac{c^4}{16\pi G}\frac{g_{ia}}{\sqrt{|g|}}\left[\left(-g\right)\left(g^{ka}g^{lb} - g^{la}g^{kb}\right)\right]_{,b}. \tag{2} $$

${}_{E}t_i^{\ k}$ are components of the canonical energy-momentum pseudotensor for gravitational field, and ${}_{F}U_i^{\ [kl]}$ is von Freud superpotential.

$$ {}_{E}K_i^{\ k} = \sqrt{|g|}\left(T_i^{\ k} + {}_{E}t_i^{\ k}\right) \tag{3} $$

are components of the *Einstein canonical energy-momentum complex for matter and gravity*.

In consequence of (1) the complex ${}_{E}K_i^{\ k}$satisfies local conservation laws

$$ {}_{E}K_i^{\ k}{}_{,k} \equiv 0. \tag{4} $$

In very special cases one can obtain reasonable integral conservation laws from these local conservation laws. Additionally, one can also introduce *the canonical superenergy tensors*. This was done in past in a series of our articles (see, e.g.,[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and references therein).

It appears that the idea of the superenergy tensors is universal: to any physical field having an energy-momentum tensor or pseudotensor one can attribute a corresponding superenergy tensor.

2 The canonical superenergy tensors

Here we give a short description of the general, constructive definition of the superenergy tensor S_a^b applicable to gravitational field and to any matter field. The definition uses *locally Minkowskian structure* of the spacetime and, therefore, it fails in a spacetime with torsion, e.g., in a Riemann-Cartan spacetime.

In the normal Riemann coordinates **NRC(P)** we define (pointwise)

$$ S_{(a)}^{\ \ (b)}(P) = S_a^{\ b} := (-)\lim_{\Omega\to P}\frac{\int\limits_{\Omega}\left[T_{(a)}^{\ \ (b)}(y) - T_{(a)}^{\ \ (b)}(P)\right]d\Omega}{1/2\int\limits_{\Omega}\sigma(P;y)d\Omega}, \tag{5} $$

where

$$ T_{(a)}^{\ \ (b)}(y) := T_i^{\ k}(y)e^i_{\ (a)}(y)e_k^{\ (b)}(y), $$
$$ T_{(a)}^{\ \ (b)}(P) := T_i^{\ k}(P)e^i_{\ (a)}(P)e_k^{\ (b)}(P) = T_a^{\ b}(P) $$

are *physical or tetrad components* of the pseudotensor or tensor field which describes an energy-momentum distribution, and $\left\{y^i\right\}$ are normal coordinates. $e^i_{\ (a)}(y)$,

$e_k^{(b)}(y)$ denote an orthonormal tetrad $e^i{}_{(a)}(P) = \delta^i_a$ and its dual $e_k^{(a)}(P) = \delta^a_k$, parallelly propagated along geodesics through P (P is the origin of the **NRC(P)**).

We have

$$e^i{}_{(a)}(y) e_i^{(b)}(y) = \delta^b_a. \tag{6}$$

For a sufficiently small 4-dimensional domain Ω which surrounds **P** we require

$$\int_\Omega y^i d\Omega = 0, \quad \int_\Omega y^i y^k d\Omega = \delta^{ik} M, \tag{7}$$

where

$$M = \int_\Omega (y^0)^2 d\Omega = \int_\Omega (y^1)^2 d\Omega = \int_\Omega (y^2)^2 d\Omega = \int_\Omega (y^3)^2 d\Omega \tag{8}$$

is a common value of the moments of inertia of the domain Ω with respect to the subspaces $y^i = 0, \ (i = 0,1,2,3)$.

As Ω we can take, e.g., a sufficiently small ball centered at P:

$$(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 \le R^2, \tag{9}$$

which for an auxiliary positive-definite metric

$$h^{ik} := 2v^i v^k - g^{ik}, \tag{10}$$

can be written in the form

$$h_{ik} y^i y^k \le R^2. \tag{11}$$

A fiducial observer **O** is at rest at the beginning **P** of the Riemann normal coordinates **NRC(P)** and its four- velocity is $v^i = * \ \delta^i_o$, where $*$ means that equation is valid only in special coordinates. In [3] $\sigma(P;y)$ denotes the two-point *world function* introduced by J.L. Synge [12]:

$$\sigma(P;y) = * \frac{1}{2}\left(y^{o^2} - y^{1^2} - y^{2^2} - y^{3^2}\right). \tag{12}$$

The world function $\sigma(P;y)$ can be defined covariantly by the *eikonal-like equation* [12]

$$g^{ik}\sigma_{,i}\sigma_{,k} = 2\sigma, \quad \sigma_{,i} := \partial_i \sigma, \tag{13}$$

together with requirements

$$\sigma(P;P) = 0, \quad \partial_i \sigma(P;P) = 0. \tag{14}$$

The ball Ω can also be given by the inequality

$$h^{ik}\sigma_{,i}\sigma_{,k} \le R^2. \tag{15}$$

Tetrad components and normal components are equal at **P**, so, we will write the components of any quantity attached to **P** without tetrad brackets, e.g., we will write $S_a{}^b(P)$ instead of $S_{(a)}{}^{(b)}(P)$ and so on.

If $T_i{}^k(y)$ are the components of an energy-momentum tensor of matter, then we get from (5)

$$ {}_mS_a{}^b(P;v^l) = \left(2\hat{v}^l\hat{v}^m - \hat{g}^{lm}\right)\nabla_l\nabla_m\hat{T}_a{}^b = \hat{h}^{lm}\nabla_l\nabla_m\hat{T}_a{}^b. \tag{16}$$

Hat over a quantity denotes its value at **P**, and ∇ means covariant derivative.

Tensor ${}_mS_a{}^b(P;v^l)$ is called *the canonical superenergy tensor for matter*.

For the gravitational field, substitution of the canonical Einstein energy-momentum pseudotensor as $T_i{}^k$ in (5) gives

$$ {}_gS_a{}^b(P;v^l) = \hat{h}^{lm}\hat{W}_a{}^b{}_{lm}, \tag{17}$$

where

$$\begin{aligned} W_a{}^b{}_{lm} = \frac{2\alpha}{9}\big[&B^b{}_{alm} + P^b{}_{alm} \\ &- \frac{1}{2}\delta^b_a R^{ijk}{}_m\left(R_{ijkl} + R_{ikjl}\right) + 2\delta^b_a\beta^2 E_{(l|g}E^g{}_{|m)} \\ &- 3\beta^2 E_{a(l|}E^b{}_{|m)} + 2\beta R^b{}_{(a|g|l)}E^g{}_m\big]. \end{aligned}$$

Here $\alpha = \frac{c^4}{16\pi G} = \frac{1}{2\beta}$, and

$$ E_i{}^k := T_i{}^k - \frac{1}{2}\delta^k_i T \tag{18}$$

is the modified energy-momentum tensor of matter [1].

On the other hand

$$ B^b{}_{alm} := 2R^{bik}{}_{(l|}R_{aik|m)} - \frac{1}{2}\delta^b_a R^{ijk}{}_l R_{ijkm} \tag{19}$$

are the components of the *Bel-Robinson tensor* (**BRT**), while

$$ P^b{}_{alm} := 2R^{bik}{}_{(l|}R_{aki|m)} - \frac{1}{2}\delta^b_a R^{jik}{}_l R_{jkim} \tag{20}$$

is the Bel-Robinson tensor with "transposed" indices (ik).

In vacuum ${}_gS_a{}^b(P;v^l)$ takes the simpler form

$$ {}_gS_a{}^b(P;v^l) = \frac{8\alpha}{9}\hat{h}^{lm}\left(\hat{C}^{bik}{}_{(l|}\hat{C}_{aik|m)} - \frac{1}{2}\delta^b_a\hat{C}^{i(kp)}{}_{(l|}\hat{C}_{ikp|m)}\right). \tag{21}$$

Here $C^a{}_{blm}$ denotes components of the *Weyl tensor*.

Some remarks:

[1] In terms of $E_i{}^k$ Einstein equations read $R_i{}^k = \beta E_i{}^k$.

1. in vacuum the quadratic form ${}_gS_a^{\ b}v^a v_b$, where $v^a v_a = 1$, *is positive-definite*. This form gives the gravitational *superenergy density* ε_g for a fiducial observer **O**.
2. In general, the canonical superenergy tensors are uniquely determined only along the world line of an observer **O**. But in special cases, e.g., in Schwarzschild spacetime or in Friedmann universes, when there exists a physically and geometrically distinguished four-velocity field $v^i(x)$, one can introduce, in an unique way, unambiguous fields ${}_gS_i^{\ k}(x; v^l)$ and ${}_mS_i^{\ k}(x; v^l)$.
3. It can be shown that the superenergy densities ε_g, ε_m, which have dimension $\frac{Joule}{(meter)^5}$, exactly correspond to the Appel's *energy of acceleration* $\frac{1}{2}\mathbf{aa}$. The Appel's energy of acceleration plays the fundamental role in Appel's approach to classical mechanics [13, 14, 15].
4. We have proposed, in our previous papers, to use the tensor ${}_gS_i^{\ k}(P; v^l)$ as gravitational energy-momentum tensor.
5. We have used the canonical superenergy tensors ${}_gS_i^{\ k}$ and ${}_mS_i^{\ k}$ to local (and also to global) analysis of some well-known solutions to the Einstein equations like Schwarzschild, Kerr, Friedmann, Gödel, Kasner, Bianchi I, de Sitter and anti-de Sitter solutions. The obtained results were interesting (see [1, 2, 3, 4, 5, 7, 8, 11]), e.g., in Gödel universes the sign of the superenergy density $\varepsilon_s := \varepsilon_g + \varepsilon_m$ depends on causality ($\varepsilon_s < 0$) and non-causality ($\varepsilon_s > 0$), and, in Schwarzschild spacetime the integral exterior superenergy S is connected with Hawking temperature T of the Schwarzschild black hole: $S = \frac{8\pi k c^3}{9\hbar G}T$. We have also studied the transformation rules for the canonical superenergy tensors under conformal rescaling of the metric $g_{ik}(x)$ [8, 16].
6. The idea of the superenergy tensors can be extended on angular momentum (see, [3, 10]). The angular supermomentum tensors *do not depend* on a radius vector and, in gravitational case, they depend only on "spinorial part" of the suitable gravitational angular momentum pseudotensor.
7. As a result of an averaging the tensors ${}_gS_a^{\ b}(P; v^l)$ and ${}_mS_a^{\ b}(P; v^l)$, in general, do not satisfy any local conservation laws. Only in a symmetric spacetime or in a spacetime which has constant curvature one can get
$$\left[{}_gS_a^{\ b}(P; v^l)\right]_{,b} = 0. \tag{22}$$
8. There exists an exchange of the canonical superenergy between gravity and matter in the following sense. Let us consider the consequence of the equations (4)
$$\left(\Delta_E^{(4)} K_i^{\ k}\right)_{,k} = \left[\left(\Delta^{(4)}(\sqrt{|g|}_E t_i^{\ k}\right) + \Delta^{(4)}\left(\sqrt{|g|}T_i^{\ k}\right)\right]_{,k} = 0, \tag{23}$$
where $\Delta^{(4)} := (\partial_0)^2 + (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2$.
The quantities (with total balance equal to zero)
$$\Delta^{(4)}\left(\sqrt{|g|}_e t_i^{\ k}\right),\ \ \Delta^{(4)}\left(\sqrt{|g|}T_i^{\ k}\right) \tag{24}$$

have dimensions of the canonical superenergy and, when taken at the origin **P** of the **NRC(P)** and written covariantly, then they coincide with the canonical superenergy tensors ${}_gS_i^{\ k}(P;v^l)$, ${}_mS_i^{\ k}(P;v^l)$ respectively.
9. Recently we have noticed that the total superenergy density is positive-definite or null for known stable solutions to the Einstein equations and negative-definite for unstable solutions. The physical meaning of this fact is under study.

Changing the constructive definition (5) to the form

$$< T_a^{\ b}(P) >:= \lim_{\varepsilon\to 0} \frac{\int\limits_{\Omega}\left[T_{(a)}^{\ \ (b)}(y) - T_{(a)}^{\ \ (b)}(P)\right]d\Omega}{\varepsilon^2/2\int\limits_{\Omega} d\Omega}, \tag{25}$$

where $\varepsilon := \frac{R}{L} > 0$ (equivalently $R = \varepsilon L$) is a real parameter and L is a dimensional constant:$[L] = m$, one obtains *the averaged relative energy-momentum tensors*. Namely, from (25) one obtains:

$$<_m T_a^{\ b}(P;v^l) >=_m S_a^{\ b}(P;v^l)\frac{L^2}{6}, \tag{26}$$

for matter and

$$<_g t_a^{\ b}(P;v^l) >=_g S_a^{\ b}(P;v^l)\frac{L^2}{6} \tag{27}$$

for gravity.

The components of the averaged relative energy-momentum tensors have correct dimensions but they depend on a dimensional parameter L. How to choose the dimensional parameter L?

In [7] we have proposed a universal choice of the parameter L. Namely, we have proposed $L = 100L_P \approx 10^{-33}m$. Here $L_P := \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35}m$ is the Planck length.

Such choice of L gives the averaged gravitational relative energy-momentum tensors components which are negligible in comparison with the components of the energy-momentum tensor for matter. Consequently, with such choice of the parameter L, these tensors play no role in evolution of the material objects and in evolution of the Universe. On the other hand, the choices can be made such that $L = \frac{2GM}{c^2}$ for a closed system with mass M, $L = \lambda$ for a gravitational wave of wave length λ, and $L = \frac{2GM_U}{c^2} = \frac{c}{H_0} = ct_0$ in cosmology, which lead us to the averaged relative energy densities of the same order as ordinary energy density of matter. Here M_U, H_0, t_0 mean mass of the observed part of the Universe, actual value of the Hubble constant and an approximate age of the Universe respectively.

Of course, there exist other possibilities of choosing the length parameter L.

3 Conclusion

On the *superenergy level*, or on the *averaged relative energy-momentum* level, there seems to be no problem with a suitable expression for gravity. However, canonical superenergy tensors seem more fundamental than the corresponding averaged relative energy-momentum tensors because they do not depend on the choice of L.

Acknowledgements

This paper was mainly supported by Polish Ministry of Science and Higher Education Grant No 505-4000-25-0976 (years 2011-1013). Author also would like to thank Professor Jiří Bičák for possibility to deliver talk during the Conference "Relativity and Gravitation – 100 years after Einstein in Prague".

References

1. J. Garecki, *Energy and superenergy of a closed universe*, Rep. Math. Phys. **33**, 57 (1993)
2. J. Garecki, *Energy-momentum and angular momentum of isotropic homogeneous cosmological models in a gauge theory of gravity*, International Journal of Theoretical Physics **34**, 2307 (1995)
3. J. Garecki, *Canonical angular supermomentum tensors in general relativity*, J. Math. Phys. **40**, 4035 (1999)
4. J. Garecki, *Do gravitational waves carry energy-momentum and angular momentum?*, Ann. Phys. (Berlin) **11**, 442 (2002)
5. M.P. Dąbrowski, J. Garecki, *Superenergy and supermomentum of Gödel universes*, Class. Quantum Grav. **19**, 1 (2002)
6. J. Garecki, *On the gravitational energy of the Bonnor spacetime*, Class. Quantum Grav. **22**, 4051 (2005)
7. J. Garecki, *The tensors of the averaged relative energy momentum and angular momentum in general relativity and some of their applications*, Found. Phys. **37**, 341 (2007)
8. M.P. Dąbrowski, J. Garecki, *Statefinders and observational measurement of superenergy*, Phys. Lett. B **686**, 6 (2010)
9. J. Garecki, *A superenergetic analysis of the plane and plane-fronted gravitational waves*, Rep. Math. Phys. **40**, 485 (1997)
10. J. Garecki, *Superenergy and angular supermomentum tensors in general relativity*, Rep. Math. Phys. **44**, 95 (1999)
11. J. Garecki, *Superenergy, conformal transformations, and Friedman universes*, Ann. Phys. (Berlin) **19**, 263 (2010)
12. J.L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960)
13. P. Appel, *Sur une forme générale des équations de la dynamique*, J. reine angew. Math. **1900**(121), 310 (1900)
14. P. Appel, *Sur une forme générale des équations de la dynamique et sur le principe de Gauss*, J. reine angew. Math. **1900**(122), 205 (1900)
15. G. Białkowski, *Classical Mechanics* (PWN, Warsaw, 1975). In Polish
16. M.P. Dąbrowski, J. Garecki, D. Blaschke, *Conformal transformations and conformal invariance in gravitation*, Ann. Phys. (Berlin) **18**, 13 (2009)