

# From ‘Nothing’ to Inflation and Back Again

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**Abstract** A procedure for solving Wheeler-DeWitt equation in Euclidean region, following step by step the construction of tunneling wave function in nonrelativistic quantum mechanics by Banks, Bender and Wu, is proposed. Solutions for a universe satisfying no-boundary condition and a universe created from ‘nothing’ are compared to the corresponding solutions for a particle in a two-dimensional potential well, and effects of indefiniteness of metric and zero energy in Wheeler-DeWitt equation are discussed.

## 1 Introduction

The basic two minisuperspace solutions of Wheeler-DeWitt equation were proposed as long ago as at the beginning of 80’s. In 1983, Hartle and Hawking introduced their no-boundary condition and used it to construct an inflationary solution in which scalar field was decoupled from gravity and its effect on the expansion of the universe was mimicked by the cosmological constant [1]. Hawking then replaced the conformal coupling of the scalar field by the mass term and obtained a truly inflationary solution [2, 3]. Since then the model has been explored repeatedly. To mention just two remarkable results, the universe was shown to avoid Big Crunch by quantum bounce for all, not just fine-tuned, initial conditions [4]; and the model was used to demonstrate the collapse of the measurement of time with the help of quantum mechanical degrees of freedom in the period of maximal expansion [5]. Shortly before the no-boundary solution appeared, in 1982, Vilenkin proposed another solution describing creation of the universe from ‘nothing’ [6, 7]. Linde advocated the use of such solution ‘in those situations where the scale parameter  $a$  itself must be quantized’ [8], and proposed a heuristic derivation of it via inverse

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Wick rotation. Later there appeared a different approach to ‘creationist’ cosmology, adopting the no-boundary condition but making use of density matrix rather than wave function [9].

Both Hawking and Vilenkin solutions contain an Euclidean region in which the wave function describes tunneling of the universe, either from a finite radius to a point (Hawking solution) or from a point to a finite radius (Vilenkin solution). In nonrelativistic quantum mechanics, a complete WKB solution in the tunneling region in two dimensions was obtained by Banks, Bender and Wu [10]. The solution refers to a particle escaping from a potential well, but with some effort can be modified to apply also to the opposite case when a particle tunnels from outside the barrier into the well. The cosmological problem is two-dimensional and allows for WKB approximation, therefore it would be natural to employ the same construction in it. However, as for now the solution by Banks, Bender and Wu was apparently not used in this way, although Vilenkin mentions in [7] that his tunneling solution is ‘similar’ to it.

In this note a procedure for computing the wave function of the universe in the Euclidean region is outlined. In section II the construction by Banks, Bender and Wu is summarized, rewritten so that it works both ways, outwards as well as inwards; in section III new features of the construction appearing when it is carried over to cosmology are discussed; and in section IV possible applications of the theory are suggested.

## 2 Tunneling in quantum mechanics

Consider a particle in two dimensions obeying Schrödinger equation

$$\left[ -\partial_x^2 - \partial_y^2 + \frac{1}{4}(x^2 + y^2) - \frac{1}{4}\varepsilon(x^4 + y^4 + 2cx^2y^2) - E \right] \psi = 0, \quad (1)$$

where the parameters  $\varepsilon$  and  $c$  satisfy  $0 < \varepsilon \ll 1$  and  $|c| < 1$ . The potential into which the particle is placed consists of a circularly symmetric well and a quatrefoil barrier around it. We are interested in two processes, tunneling of the particle from the well to the other side of the barrier and *vice versa*, both along the positive  $x$  axis. The particle tunneling outwards has a (quasi)discrete spectrum, which coincides to a great precision, if the energies are not too high, with the spectrum in a well extrapolated to infinity. From the requirement that only an outgoing wave appears behind the barrier it follows that the energies acquire a small imaginary part equal to  $-\frac{1}{2} \times$  the decay rate of the state inside the well. The particle tunneling inwards, on the other hand, can have any energy. In what follows we assume that  $\psi$  separates in variables  $x, y$  inside the well and the energy  $E_y$  going to the  $y$  direction equals  $\frac{1}{2}$ . In particular, the particle tunneling outwards can be in the ground state with  $E = 1$  and  $E_x = E_y = \frac{1}{2}$ .

Let us start with rewriting the potential near the  $x$  axis as

$$V = V_0 + \frac{1}{4}ky^2, \quad (2)$$

where

$$V_0 = \frac{1}{4}x^2(1 - \epsilon x^2), \quad k = 1 - 2\epsilon cx^2. \quad (3)$$

The idea of solving (1) in the tunneling region is to separate a WKB wave function into the longitudinal direction and a bell-shaped function with a variable width in the transversal direction out of  $\psi$ . Thus, we write

$$\psi = Ap^{-1/2} \exp\left(\mp \int p dx - \frac{1}{4}fy^2\right), \quad p = \sqrt{V_0 - E_x}, \quad (4)$$

with  $f$  depending on  $x$  and  $A$  depending on both  $x$  and  $y$ . The upper and lower signs refer to a particle tunneling outwards and inwards respectively. Equation for  $f$  is obtained by collecting the terms in the original equation proportional to  $y^2$  and putting them equal to zero. In this way we relate the function  $f$ , defining the width of the wave function, to the function  $k$ , defining the width of the valley the particle is tunneling through; and we obtain an equation for  $A$  that separates after  $y$  is replaced by an appropriate variable proportional to it.

In the leading order of WKB approximation equations for  $f$  and  $A$  lose the second derivative with respect to  $x$  and, in addition, the function  $p$  multiplying the first derivative loses the constant  $E_x$  under the square root sign. After rescaling  $x \rightarrow \epsilon^{-1/2}x$  we have

$$2p \frac{d}{dx} \rightarrow xw \frac{d}{dx} = -x^2 \frac{d}{dw}, \quad w = \sqrt{1 - x^2},$$

and equation for  $f$  takes the form

$$\pm x^2 \frac{df}{dw} = f^2 - k, \quad k = 1 - 2cx^2. \quad (5)$$

The equation can be linearized by introducing an auxiliary function  $u$  such that

$$f = \mp \frac{x^2}{u} \frac{du}{dw}. \quad (6)$$

After inserting this into (5) we find that  $u$  obeys equation for associated Legendre function with the lower index given by

$$v(v+1) = 2c \quad (7)$$

and the upper index equal to  $\pm 1$ . Equation for  $A$  separates in the variables  $w$  and  $s = y/u$  and if we write  $A$  as  $WS$ , where  $W$  is a function of  $w$  and  $S$  is a function of  $s$ , we find that  $S$  is either cosine or hyperbolic cosine or constant. Matching the tunneling solution to the solution inside the well singles out the third possibility. Thus,  $A$  depends on  $w$  only, and after a little algebra one finds

$$A = \text{const} \left( \frac{1-w}{1+w} \right)^{\pm 1/4} u^{-1/2}. \quad (8)$$

Let us examine the behavior of  $f$ . To simplify the analysis, we return from the variable  $w$  to  $x$  with the help of the formula

$$x \frac{d}{dw} = -w \frac{d}{dx}.$$

First we find the asymptotic of  $f$  for  $x \sim 0$ . In order that the internal and external solutions match,  $f$  must equal 1 at  $x = 0$ ; thus, we write  $f$  as  $1 + \Delta$  and skip the  $\Delta^2$  term in (5). We obtain the equation

$$\mp x \frac{d\Delta}{dx} = 2(\Delta + cx^2), \quad (9)$$

whose solution is

$$\Delta = \begin{cases} -\frac{1}{2}cx^2 \\ 2cx^2(\log x + \mathcal{C}) \end{cases} \quad (10)$$

where the upper and lower expression refer to the upper and lower sign in (9) and  $\mathcal{C}$  is integration constant. We can see that the width of the tunneling wave function is given uniquely for a particle tunneling outwards, but the strips on which the wave function is nonzero form a one-parametric sequence for a particle tunneling inwards. The reason is presumably that the waves incident on the barrier along the positive  $x$  axis form a one-parametric sequence, too, and different waves tunnel along different stripes.

From the asymptotic of  $f$  we can determine  $u$ . For  $w \sim 1$ , or equivalently,  $x \doteq \sqrt{2(1-w)} \sim 0$ , the asymptotics of the associated Legendre functions of first and second kind are

$$P_v^{-1} = \frac{1}{2}x, \quad Q_v^1 = -x^{-1}.$$

(It suffices to consider one upper index in  $P_v$  and  $Q_v$  since the functions with opposite upper indices are proportional to each other.) To obtain  $f = 1$  for  $x = 0$  we must choose  $u$  equal to  $P_v^{-1}$  and to  $Q_v^1 + \text{some coefficient} \times P_v^{-1}$  for a particle tunneling out- and inwards. Moreover, by including the next-to-leading term into the expansion of  $Q_v^1$ ,

$$\Delta Q_v^1 = \frac{1}{4}v(v+1)[2\log \frac{x}{2} + \psi(v) + \psi(v+2) - \psi(2) + \gamma]x + \frac{1}{4}x,$$

where  $\psi$  is digamma function and  $\gamma$  is Euler-Mascheroni constant, we can find the coefficient in front of  $P_v^{-1}$  in the latter case.

Finally, we can use  $u$  to compute  $f$  on the whole interval of  $x$ . Note that for  $c > \frac{1}{2}$  the function  $k$  becomes negative before  $x$  reaches 1, so that the valley the particle is tunneling through turns to a slope. According to (5), this drives the derivative  $df/dx$  to more negative values and makes  $f$  to fall down faster for a particle tunneling outwards. Nevertheless,  $f$  stays positive and  $\psi$  stays suppressed in the transversal direction up to  $x = 1$  for any  $c$ . If the particle is tunneling inwards,  $f$  increases near

$x = 1$  for the values of  $c$  in question, and only with decreasing  $c$  it starts to decrease. For low enough  $c$ 's  $f$  passes through zero before  $x$  reaches 1; however, this holds only for  $\mathcal{C} = 0$  and can be cured by choosing positive  $\mathcal{C}$ .

### 3 Tunneling in cosmology

Consider a closed universe with a scalar field living in it, and denote the radius of the universe by  $a$  and the value of the field by  $\phi$ . Suppose, furthermore, that the field is massive with the mass  $m$  and the theory includes cosmological constant  $\Lambda$ ; thus, the field has quadratic potential shifted upwards by  $\lambda = \frac{1}{3}\Lambda$ . The wave function of the universe satisfies the Wheeler-DeWitt equation

$$[-\partial_a^2 + a^{-2}\partial_\phi^2 + a^2(1 - \lambda a^2 - m^2 a^2 \phi^2)]\psi = 0. \quad (11)$$

To account for the ambiguity due to operator ordering, the operator  $-\partial_a^2$  has to be modified to  $-a^{-\mu}\partial_a a^\mu \partial_a$  with an arbitrary  $\mu$ . However, in the WKB approximation we are interested in this affects only the preexponential factor in  $\psi$ , therefore we can put  $\mu = 0$ .

Equation (11) is almost identical to the equation we obtain from (1) by passing from rectangular to polar coordinates, restricting the polar angle to  $|\phi| \ll 1$  and replacing the operator  $-r^{-1}\partial_r r \partial_r$ , again with the reference to WKB approximation, by  $-\partial_r^2$ . The new equation reads

$$[-\partial_r^2 - r^{-2}\partial_\phi^2 + \frac{1}{4}r^2(1 - \varepsilon r^2 + 2\varepsilon\gamma r^2 \phi^2) - E]\psi = 0, \quad (12)$$

where  $\gamma = 1 - c$ . We will solve this equation in a similar way as equation (1) and discuss equation (11) later. First we express  $\psi$  as

$$\psi = Bq^{-1/2} \exp\left(\mp \int q dr - \frac{1}{4}g\phi^2\right), \quad q = \sqrt{\mathcal{V}_0 - E}, \quad (13)$$

with  $\mathcal{V}_0 = \frac{1}{4}r^2(1 - \varepsilon r^2)$  and  $g$  related to  $\kappa = 2\varepsilon\gamma r^4$  in a similar way as  $f$  was related to  $k$ . After rescaling  $r \rightarrow \varepsilon^{-1/2}r$ ,  $\phi \rightarrow \varepsilon^{1/2}\phi$ ,  $\kappa \rightarrow \varepsilon^{-1}\kappa$  and  $g \rightarrow \varepsilon^{-1}g$  and introducing  $\xi = \sqrt{1 - r^2}$  we have

$$\pm r^2 \frac{dg}{d\xi} = r^{-2}g^2 - \kappa, \quad \kappa = 2\gamma r^4, \quad (14)$$

and after writing  $g$  as

$$g = \mp \frac{r^4}{v} \frac{dv}{d\xi}, \quad (15)$$

we find that  $v$  obeys equation for Gegenbauer function with the lower index given by

$$\alpha(\alpha + 3) = -2\gamma \quad (16)$$

and the upper index equal to  $\frac{3}{2}$ . Finally, for the function  $B$  we obtain

$$B = \text{const } v^{-1/2}. \quad (17)$$

To fix the combination of Gegenbauer functions in  $v$  we must explore the behavior of  $g$  for  $r \sim 0$ . When doing so we notice that now there are *two* terms proportional to  $y^2$  in the exponential near the origin, the term  $-\frac{1}{4}g\varphi^2 \doteq -\frac{1}{4}r^{-2}gy^2$  and the term  $\mp\frac{1}{4}y^2$  coming from

$$\int q d\hat{r} \sim \int \frac{1}{2}\hat{r}d\hat{r} = \frac{1}{4}\hat{r}^2 = \frac{1}{4}(\hat{x}^2 + \hat{y}^2),$$

where we have denoted the original, non-rescaled variables  $x$  and  $r$  by  $\hat{x}$  and  $\hat{r}$ . These two terms must add to produce the term  $-\frac{1}{4}y^2$  appearing in the solution inside the well. The resulting asymptotics of  $g$  are

$$g = \begin{cases} \frac{1}{2}\gamma r^4 \\ 2r^2[1 + cr^2(\log r + \mathcal{D})] \end{cases} \quad (18)$$

This yields  $v$  equal to  $C_\alpha^{3/2}$  and  $D_\alpha^{3/2} + \text{some coefficient} \times C_\alpha^{3/2}$  for a particle tunneling out- and inwards respectively; and knowing  $v$ , we can determine the behavior of  $g$  on the whole interval of  $r$ .

Once we have found  $f$  we do not need to compute  $g$  from scratch. Instead, we can express  $g$  in terms of  $f$ . For that purpose we insert

$$\int q d\hat{r} = \int \frac{1}{2}\hat{r}\xi d\hat{r} = \varepsilon^{-1} \int \frac{1}{2}r\xi dr = -\varepsilon^{-1}\frac{1}{6}(\xi^3 - 1) \doteq \int p d\hat{x} + \frac{1}{4}\xi y^2$$

into (13) and compare the resulting expression to (4). We obtain

$$g = r^2(f \mp \xi), \quad (19)$$

where  $f$  is to be regarded as a function of  $r$ . This coincides with the function  $g$  constructed previously, if we express  $C_\alpha^{3/2}$  and  $D_\alpha^{3/2}$  in terms of  $P_v^{-1}$  and  $Q_v^1$  and put  $\mathcal{D} = \mathcal{C} - \frac{1}{4}$ .

Equation (11) differs from (12) in that it has reversed sign of the kinetic term in  $\phi$  direction and vanishing energy. Because of the former property the metric of the kinetic term is indefinite. The sign of the mass term is reversed or stays the same depending on whether  $m$  is real, which corresponds to a metastable vacuum, or imaginary, which corresponds to an unstable vacuum. However, it is the *relative* signs with respect to the kinetic terms in  $a$  and  $\phi$  directions which matter; and no matter what the absolute sign, one of these signs stays the same while the other is reversed.

The two solutions of equation (11), with minus and plus sign in the exponential, describe tunneling of the universe outwards, from a point to a finite radius, and inwards, from a finite radius to a point. An immediate consequence of the indefiniteness

of metric is that the signs in the equation for  $g$  are switched. As a result, we obtain one-parametric class of  $g$ 's for a universe tunneling outwards and a single  $g$  for a universe tunneling inwards. Having just one solution in the latter case is consistent with the observation that the wave function of the universe is determined completely by the no-boundary condition. Having infinitely many solutions in the former case can be explained by the fact that the energy is zero, which means that the imaginary part of energy is zero, which means that the outgoing probability current stays finite up to the zero radius of the universe. The corresponding outgoing waves must be put into the theory by hand and apparently form a one-parametric sequence, similarly as ingoing waves in the quantum mechanical problem with a particle tunneling inwards.

In the previous discussion we have assumed that  $m$  is real. However, as mentioned by Vilenkin in [7], when considering a universe tunneling outwards it is reasonable to pass to imaginary  $m$ . The point is that the tunneling path shortens, and the wave function becomes less suppressed behind the barrier, if the shift of the potential  $\lambda$  increases. Thus, the tunneling is most effective at the global maximum of the potential. Imaginary  $m$  and tunneling outwards in cosmology corresponds to  $c > 1$  and tunneling inwards in quantum mechanics. For such tunneling we find, in addition to a one-parametric class of  $g$ 's with the asymptotic given in the lower line of (18), one more  $g$  with the asymptotic

$$g = -\frac{1}{2}\gamma r^4. \quad (20)$$

This solution must be abandoned since the corresponding  $f$  equals  $-1$  at  $x = 0$ , which means that the wave function explodes in the transversal direction. However, in the cosmological setting such  $g$  seems admissible, and even privileged because of its one-to-one correspondence with  $g$  appearing in the problem with a universe tunneling inwards.

## 4 Conclusion

We have shown how the procedure by Banks, Bender and Wu can be carried over to cosmology and used to construct the wave function of the universe in the Euclidean region. The way how to do that has been only sketched here, the details will be given elsewhere [11]. In particular, we have skipped the discussion of the behavior of the wave function ‘inside the well’, in the region where the radius of the universe is close to zero. This question is vital for the construction, because as long as the tunneling solution is not matched with the solution ‘inside the well’, it remains unjustified.

For a universe tunneling inwards, tunneling solution converts at the edges of Euclidean region into oscillatory one, describing time-symmetric evolution during which the universe repeatedly crosses that region. The crossings do not seem to change the course of the evolution, but can still have some imprint on it, and the knowledge of the exact form of tunneling solution can help to identify that imprint. For a universe tunneling outwards there apparently exists, in addition to ‘regular’

solution which behaves like that for a universe tunneling inwards, a one-parametric class of solutions with markedly different behavior. Such solutions can mediate tunneling not only to the maximum of potential but also to its minimum, therefore can be helpful when contemplating the possibility of creating a universe with scalar field in metastable state directly from ‘nothing’.

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