Exact Hairy Black Holes

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Abstract This contribution reviews the recent discovery of a certain class of – regular on and outside the horizon – exact hairy black hole solutions in four dimensional general relativity. Their construction follows from the integrability of a cohomogeneity two Weyl rescaling of the Carter-Debever ansatz in the presence of an arbitrary number of scalar fields with an arbitrary self interaction and an arbitrary non-minimal coupling to the scalar curvature. Two field equations, independent of the specific form of the energy momentum tensor, are used to integrate the metric. The remaining ones fix the form of the scalar field self interaction. The cohomogeneity one black holes are described and are shown to encompass all the exact – regular in the domain of outer communications – uncharged, black holes with a minimally coupled scalar hair, available in the literature.

1 Introduction and Discussion

The field of exact solutions in gravity, as well as their interpretation, is as old as general relativity and the research group at Charles university, and their collaborators, are well known for their contributions to this subject. Many of them can be found in the review [1] or the book [2]. From the black hole uniqueness theorems it is already well known that at least in four dimensions, the asymptotically flat, stationary and regular black holes in the electrovacuum case are exhausted, for references see [3]. Therefore, it is natural to attempt to extend these studies when other matter fields are included. Indeed, the studies of the minimally coupled scalar field have a prominent role in the construction of black holes. In the static, asymptotically flat case, the minimally coupled no-hair conjecture was shown to be true for convex potentials [4], and, more generally, for potentials satisfying the strong [5] and weak

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energy condition [6] . These studies have their counterpart in Brans-Dicke [7] and more generally in scalar-tensor theories [8], showing that whenever the scalar field potential satisfies the weak energy condition in the Einstein frame and the black hole spacetime is stationary and asymptotically flat it must be Kerr. When the scalar field satisfies the null energy condition an exact family of spherically symmetric black hole solutions has been recently constructed [9].

When the cosmological constant is negative exact uncharged AdS_4 hairy black hole solutions have been extensively studied [10, 11, 12, 13, 14]. There is a precise conjecture on the non-existence of spherically symmetric black holes in AdS for scalar field potentials that comes from "the right" superpotential [15]. These solutions are interesting in the light of the AdS/CFT conjecture. In particular, in four dimensions, and when the scalar field is charged, they define the setting for the AdS/Condensed matter correspondence [16]. When the cosmological constant is positive the black holes have also attracted some attention of the community [17].

This article intends to shortly summarize my recent contributions to the subject. I have followed the idea that stationary and axisymmetric spacetimes that have a hidden symmetry, in the form of a conformal Killing tensor, should allow for a complete integrability of some form of a non-trivial self interaction of the scalar field. Therefore, in [14] I explicitly showed that, starting with the ansatz that contains all the vacuum Petrov type D solutions, it is possible to integrate the system in the presence of a non-minimally coupled scalar field or a non-linear sigma model. It is very interesting to note that the self interaction of the scalar field is completely fixed by the form of the metric ansatz and, therefore, the scalar field potential is an output of the analysis. While these results are presented in the Einstein frame, their extension to a Scalar-Tensor theories in some Jordan frame or $F(R)$ theory is straightforward.

The scalar field potential turns out to be contained as special case of all the exact hairy (A)dS black holes available in the literature. The static solutions are black holes continuously connected with the Schwarzschild (A)dS solution, and can be generalized to include non-minimally coupled gauge fields [18]. In asymptotically AdS black holes, with cosmological constant $\Lambda = -3/l^2$, the scalar field mass is $m^2 = -2/l^2$, which is above the Breitenlohner-Freedman bound, $m^2 = -9/4l^2$, ensuring the perturbative stability of these black holes. This mass is the one of the scalar fields of the $U(1)^4$ truncation of gauged $N = 8$ supergravity [19] and the solutions can be embedded in this supergravity theory.

The content of the article is as follows. In the second section the general integrability of the ansatz with two Killing vectors is reviewed and in the third section the static case and its special limits are presented.

2 The integrable system with two Killing vectors

The conventions are given by the action principle

$$S(g,\phi)=\int d^4x\sqrt{-g}\left[\frac{R}{2\kappa}-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{\xi}{12}\phi^2R-V(\phi)\right], \tag{1}$$

where $\kappa=8\pi G$. We are interested in studying a cohomogeneity two Weyl rescaling of the Carter-Debever [20, 21], also studied by Plebański [22]:

$$ds^2=S(q,p)\Big(\frac{1+p^2q^2}{Y(q)}dq^2+\frac{1+p^2q^2}{X(p)}dp^2-\frac{Y(q)}{1+p^2q^2}\left(p^2d\tau+d\sigma\right)^2 + \frac{X(p)}{1+p^2q^2}\left(d\tau-q^2d\sigma\right)^2\Big) \tag{2}$$

When $S(q,p)=p^{-2}$, this metric contains the Kerr-Newman hole with a cosmological constant. Letting $S(q,p)$ free, this metric can be integrated in vacuum, and with the same Maxwell field like in the Kerr-Newman case; the Plebański-Demiański spacetime arises [23].

The observation made in [14], is that for stationary and axisymmetric scalar fields, $\phi=\phi(q,p)$, the energy momentum tensor of a scalar field

$$T_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}g_{\mu\nu}\left(\partial\phi\right)^2-g_{\mu\nu}V(\phi), \tag{3}$$

is such that components $T^\tau_\sigma=0=T^\sigma_\tau$ and, therefore, the Einstein equations, $R^\mu_\nu-\frac{1}{2}\delta^\mu_\nu R=\kappa T^\mu_\nu$, imply $R^\tau_\sigma=0=R^\sigma_\tau$. These two equations are enough to complete the metric functions; the solution is

$$X(p)=C_0+C_2p^2+C_4p^4+C_1p^{-\nu+2}+C_3B_3p^{\nu+2}, \tag{4}$$
$$Y(q)=C_4-C_2q^2+C_0q^4+C_3C_1q^{-\nu+2}+B_3q^{\nu+2}, \tag{5}$$

$$S(q,p)=C\frac{p^{\nu-1}q^{\nu-1}}{(C_3p^\nu+q^\nu)^2}. \tag{6}$$

This solution reduces to the Plebański-Demiański spacetime when $\nu=\pm1$. The remaining Einstein equations fix the scalar field and the scalar field potential to a very precise form. The same process can be done when the scalar field is non-minimally coupled to gravity and, more generally, when a non-linear sigma model is the source of the Einstein equations.

To extract more physical information let us study the cohomogeneity one black holes.

3 The static black holes

The static limit of the previous configuration is

$$ds^2 = \Omega(r)(-F(r)dt^2 + \frac{dr^2}{F(r)} + d\Sigma^2), \tag{7}$$

$$\Omega(r) = \frac{\nu^2 \eta^{\nu-1} r^{\nu-1}}{(r^\nu - \eta^\nu)^2}, \qquad \phi = l_\nu^{-1} \ln(r\eta^{-1}), \tag{8}$$

$$F(r) = \frac{r^{2-\nu}\eta^{-\nu}(r^\nu - \eta^\nu)^2 k}{\nu^2} + \left(\frac{1}{(\nu^2-4)} - \left(1 + \frac{\eta^\nu r^{-\nu}}{\nu-2} - \frac{\eta^{-\nu} r^\nu}{\nu+2}\right)\frac{r^2}{\eta^2\nu^2}\right)\alpha - \frac{\Lambda}{3}, \tag{9}$$

where $l_\nu = \left(\frac{2\kappa}{\nu^2-1}\right)^{\frac{1}{2}}$ and $d\Sigma^2$ is the line element of a surface of constant curvature k . η is the only integration constant of the black hole. The solution and theory are invariant under the transformation $\nu \to -\nu$.

The scalar field potential is

$$V(\phi) = \frac{\Lambda(\nu^2-4)}{6\kappa\nu^2}\left(\frac{\nu-1}{\nu+2}e^{-(\nu+1)\phi l_\nu} + \frac{\nu+1}{\nu-2}e^{(\nu-1)\phi l_\nu} + 4\frac{\nu^2-1}{\nu^2-4}e^{-\phi l_\nu}\right) + \frac{\alpha}{\nu^2\kappa}\left(\frac{\nu-1}{\nu+2}\sinh((1+\nu)\phi l_\nu) + \frac{\nu+1}{\nu-2}\sinh((1-\nu)\phi l_\nu) + 4\frac{\nu^2-1}{\nu^2-4}\sinh(\phi l_\nu)\right). \tag{10}$$

It is easy to see from the form of the metric, and without any reference to the details of the solution itself, that it is possible to introduce Eddington-Finkelstein coordinates $u_\mp = t \pm \int \frac{dr}{F(r)}$, which allow to cover either the black hole (u_-) or the white hole (u_+). The asymptotically flat solution has a single horizon from which it follows that the Penrose diagram is the same as for the Schwarzschild black hole.

The energy momentum of the scalar field, in a comoving tetrad, has the form $T^{ab} = \mathrm{diag}(\rho, p_1, p_2, p_2)$ and, in the static regions of the spacetime, defined by $F(r) > 0$, satisfies the null energy condition

$$\rho + p_2 = 0, \qquad \rho + p_1 = \frac{(\nu^2-1)(r^\nu - \eta^\nu)^2 F(r)}{2r\nu^2\eta^{\nu-1}r^\nu} > 0. \tag{11}$$

In the hairless limit, $\nu = 1$, the change of coordinates $r = \eta - \frac{1}{y}$ brings the hairy solution (7)–(9) to the familiar Schwarzschild–de Sitter black hole

$$ds^2 = -(k - \frac{2M}{y} - \frac{\Lambda}{3}y^2)dt^2 + \frac{dy^2}{k - \frac{2M}{y} - \frac{\Lambda}{3}y^2} + y^2 d\Sigma. \tag{12}$$

where $M=\frac{3\eta^2k+\alpha}{6\eta^3}$.

The parameterization of the black holes has been chosen such that the leading order at $r=\eta$ is either Minkowski, anti–de Sitter or de Sitter in the following form

$$ds^2_{r=\eta}=\frac{1}{(r-\eta)^2}\left(-\left(k(r-\eta)^2+\frac{\Lambda}{3}\right)dt^2+\frac{dr^2}{(k(r-\eta)^2+\frac{\Lambda}{3})}+d\Sigma^2\right). \quad (13)$$

The easiest way to see that there is always an α such that $F(r)$ has a simple zero is to see that the equation $F(r_+)=0$ is linear in α

$$0=\frac{r_+^{2-\nu}\eta^{-\nu}\left(r_+^\nu-\eta^\nu\right)^2k}{\nu^2} + \left(\frac{1}{(\nu^2-4)}-\left(1+\frac{\eta^\nu r_+^{-\nu}}{\nu-2}-\frac{\eta^{-\nu}r_+^\nu}{\nu+2}\right)\frac{r_+^2}{\eta^2\nu^2}\right)\alpha-\frac{\Lambda}{3}, \quad (14)$$

therefore it is possible to solve this equation for α for any value of the other parameters.

As a final remark it is instructive to compare the behaviour of these solutions in AdS, with the asymptotic form given in [24] . When the backreaction is ignored, a scalar field with mass m minimally coupled to an AdS background has the well known fall–off $\phi\sim\frac{a}{\rho^{\Delta_-}}+\frac{b}{\rho^{\Delta_+}}$ where $\Delta_\pm$ are the roots of $\Delta(3-\Delta)+m^2l^2=0$. When $-\frac{9}{4l^2}\le m^2<-\frac{5}{4l^2}$, both branches are normalizable but the a branch contributes to the surface charges of the system. The form of the potential makes it possible to see that the mass is $m^2=-\frac{2}{l^2}$. When the mass is exactly $\frac{\Delta_+}{\Delta_-}=2$ then the scalar field develops a logarithmic branch that, again, has a non-trivial contribution to the charges at infinity. However this logarithmic branch only appears if the expansion of the potential contains a cubic term. Indeed, it is possible to verify that with the change of coordinates $r=\eta\exp(\frac{1}{\eta\rho}-\frac{1}{2\rho^2\eta^2}-\frac{\nu^2-9}{24\eta^3\rho^3})$, the scalar field takes the form

$$\phi=l_\nu^{-1}\left(\frac{1}{\eta\rho}-\frac{1}{2\rho^2\eta^2}-\frac{\nu^2-9}{24\eta^3\rho^3}\right), \quad (15)$$

and the departure from the AdS metric, defined by
$ds^2=-(k+\frac{\rho^2}{l^2})dt^2+\left(k+\frac{\rho^2}{l^2}\right)^{-1}d\rho^2+\rho^2d\Sigma$, is

$$h_{mn}=\frac{\nu^2-4}{6\eta^3\rho}g_{mn}+O(\rho^{-2}), \quad (16)$$

$$h_{tt}=\frac{\Lambda(\nu^2-4)}{18\eta^3\rho}+\frac{k(\nu-1)+6M\nu^2\eta^\nu}{3\eta\rho}+O(\rho^{-2}), \quad (17)$$

$$h_{\rho\rho} = \frac{3\left(\nu^2-1\right)}{4\eta^2\Lambda\rho^4} + O(\rho^{-5}), \tag{18}$$

where g_{mn} are the components along $d\Sigma$. This coincides exactly with (6.2) of [24] with $\Delta_- = \Delta = 1$, $a = \frac{1}{\eta l_\nu}$ and $b = -\frac{1}{2\eta^2 l_\nu}$. The case $\nu^2 = 4$ is peculiar in the sense that the deformation of the metric at infinity is subleading as for generic ν.

The cases with $\nu = 2$ and $\nu = \infty$ are special, and can be treated by a simple limiting procedure.

3.1 The case $\nu = 2$

Indeed, the potential (10) has a smooth limit when $\nu = 2$, which is given by

$$V(\phi) = \frac{\alpha}{16\kappa}\left(\sinh(3\phi l_2) + 9\sinh(\phi l_2) - 12\phi l_2 \cosh(\phi l_2)\right) + \frac{\Lambda}{2\kappa\nu^2}\left(e^{\phi l_2} + e^{-\phi l_2}\right). \tag{19}$$

where $l_2 = \sqrt{\frac{2\kappa}{3}}$. The metric functions also have a smooth limit

$$\Omega(r) = \frac{4\eta r}{\left(r^2-\eta^2\right)^2}, \tag{20}$$

$$F(r) = \frac{\eta^{-2}\left(r^2-\eta^2\right)^2}{4}k + \left(\frac{3}{16} + \left(\frac{r}{2\eta}\right)^4 - \left(\frac{r}{2\eta}\right)^2 + \frac{1}{4}\ln\left(\frac{r}{\eta}\right)\right)\alpha - \frac{\Lambda}{3}. \tag{21}$$

The potential (10) has been considered in the context of the existence of topological AdS black holes in [11]. When $\alpha = 0$ and $k = -1$ this is the MTZ black hole [10].

3.2 The case $\nu = \infty$

The $\nu = \infty$ case is a bit more subtle. First, it is necessary to rescale the area of the unit sphere as $d\Sigma \longrightarrow \nu^{-2} d\Sigma$ which implies that the metric function F rescales accordingly

$$F(r) = r^{2-\nu}\eta^{-\nu}\left(r^\nu - \eta^\nu\right)^2 k + \left(\frac{1}{(\nu^2-4)} - \left(1 + \frac{\eta^\nu r^{-\nu}}{\nu-2} - \frac{\eta^{-\nu} r^\nu}{\nu+2}\right)\frac{r^2}{\eta^2\nu^2}\right)\alpha, \tag{22}$$

and the solution is now

$$ds^2 = \Omega(r)(-F(r)dt^2 + \frac{dr^2}{F(r)} + \nu^{-2}d\Sigma^2), \tag{23}$$

$$\Omega(r) = \frac{\nu^2\eta^{\nu-1}r^{\nu-1}}{(r^\nu - \eta^\nu)^2}, \qquad \phi = l_\nu^{-1}\ln(r\eta^{-1}). \tag{24}$$

Let us introduce the changes of coordinates $r = \rho^{\frac{1}{\nu}}$, $t = \frac{\tau}{\nu}$, and the reparameterization $\eta \to \eta^{\frac{1}{\nu}}$, $\alpha \longrightarrow \nu^3\alpha$. The $\nu = \infty$ limit is then easily seen to give

$$ds^2 = \Omega_\infty(\rho)(-F_\infty(\rho)d\tau^2 + \frac{d\rho^2}{F_\infty(\rho)} + d\Sigma^2), \tag{25}$$

$$\Omega_\infty(\rho) = \frac{\eta\rho}{(\rho-\eta)^2}, \qquad \phi = \frac{1}{\sqrt{2\kappa}}\ln(\rho\eta^{-1}), \tag{26}$$

$$F_\infty(\rho) = \rho^{-1}\eta^{-1}(\rho-\eta)^2 k + \left(2\ln(\frac{\eta}{\rho}) + \frac{\rho}{\eta} - \frac{\eta}{\rho}\right)\alpha - \frac{\Lambda}{3}, \tag{27}$$

$$V_\infty(\phi) = \frac{2\alpha}{\kappa}\left(2\phi l_P + \phi l_P \cosh(\phi l_P) - 3\sinh(\phi l_P)\right) + \frac{\Lambda}{3}\left(4 + 2\cosh(l_P\phi)\right), \tag{28}$$

where $l_P = \sqrt{2\kappa}$ is proportional to the Planck length. The potential (28) plus the corresponding limit of the part proportional to Λ of (10) was considered in the context of de Sitter hairy black holes compatible with inflation in [17].

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