Classical and Quantum Scattering in Impulsive Backgrounds

Peter Aichelburg and Herbert Balasin

Abstract We present a simple but rigorous approach to solve the nonlinear-distributional scattering problem of particles in impulsive fields. As an illustration we consider a Dirac particle in an impulsive Yang-Mills wave.

1 Introduction

Due to the singular nature of the pulse and the nonlinear character of the problem a generalized framework for singular functions (distributions) has to be used. This will be done in terms of Colombeau's new generalized functions, which form an algebra therefore allowing for nonlinear operations and contain distribution space via an appropriate coarse-graining operation (association) [1]. From the physical point of view the Colombeau algebra affords a systematic way to deal with "regularizations" of singular objects, i.e. idealized situations that contain only certain parts of the information (i.e. total charge in case of point-charge distribution) whereas the other regularization-specific information is discarded. In this regard Colombeau objects contain information upon the small-scale (micro-)structure of the physical objects. Upon nonlinear operations part of the micro-structure can be magnified to the macroscopic level. At this level equality is modelled by the so-called association operation which groups together (different) Colombeau-objects that contain the same macro-aspect. However, this coarse-graining operation is in general incompatible with non-linear operations due to the aforementioned magnification effect. The most well-known example of this effect being the product of arbitrary powers of the

Peter Aichelburg
Gravitationsphysik Universität Wien

Herbert Balasin
Institut für Theoretische Physik, TU-Wien

[1] We are going to use the so-called simplified algebra [1] which allows a concise treatment and has the advantage of being immediately usable in the manifold context [2].

Heaviside-θ with θ', i.e. $\theta^{n-1}\theta' \approx (1/n)\delta$. Although we have $\theta^n \approx \theta$ and $\theta' \approx \delta$, that is to say although all powers of θ have the same macro-aspect as θ itself and θ' has the same macro-aspect as δ the product $\theta^n\theta'$ nevertheless remembers the microaspect of which power of θ has been multiplied by θ' the resulting object having the same macro-aspect as δ but different pre-factors.

2 The method

Our approach to solve the non-linear scattering problems employs precisely this structural interplay between macroscopic and microscopic aspects. Namely, in the first step we construct the solution from the macroscopic form of the scattering-equation, i.e. by gluing free solutions. The gluing of the solution and the singular character of the equations contain the microscopic information. Imposing the nonlinear scattering equation we obtain upon coarse graining, in the second step, magnified microaspects which appear in form of undetermined parameters. In order to determine these ambiguities we invoke additional conditions in the form of conservation-laws that would be an automatic consequence in the smooth context but do not necessarily follow in the singular context, since they involve nonlinear operations. In this sense we let the equations themselves determine the ambiguities and do not stipulate their values beforehand. Our approach does, due to the use of the Colombeau-algebra, not rely on any particular regularization scheme but rather determines the conditions that physically sensible regularizations have to obey. At the same time keeping most of the simple set-up of the original problem. The scattering by impulsive fields initiated by 't Hooft [3] using Penrose's [4] cut-and-paste method has been studied previously in a number of papers e.g. the classical aspects of test particles (without spin) in gravitational pp-waves first by Balasin [5], and then by Kunzinger and Steinbauer [6], making use of the full Colombeau machinery. The scattering of Dirac particles was considered e.g. by Sanchez and deVega [7], however with the drawback of an arbitrary choice of the arising ambiguities.

3 Impulsive Yang-Mills scattering

In the following we will consider scattering of quantum particles in an impulsive pp-wave Yang-Mills field. The latter is treated in analogy to the gravitational case

$$A^i_a = f^i p_a \quad (p\partial)f^i = 0 \; p^2 = 0 \; \partial_a p^b = 0.$$

Borrowing terminology from the gravitational case the Lie-algebra-valued function f^i will be referred to as the wave-profile. For the quantum particle we will consider the corresponding Dirac-equation with respect to a pp-wave Yang-Mills field, where we take the spinor-field to transform with respect to the canonical representation of the Lie-group G on its Lie-algebra

$$(\gamma^a D_a - m)\psi = 0 \qquad \{\gamma_a, \gamma_b\} = 2\eta_{ab} \qquad D_a = \partial_a - [A_a, .].$$

Taking the pp-wave character of the Yang-Mills field into account the Dirac equation becomes

$$(\gamma^a \partial_a - m)\psi - [f, \not{p}\psi] = 0.$$

The impulsive nature of the wave reflects itself in the appearance of a delta function in the wave profile concentrated on the null-hyperplane with generator p^a, i.e.$f^i = \delta(px)\tilde{f}^i$ where $\tilde{f}^i$ is the reduced profile that depends only on the spacelike directions along the null-plane. Within the Colombeau-algebra we may glue the Dirac-spinor ψ from two solutions of the free Dirac-equation "above" and "below" the pulse-plane

$$\psi = \theta_+ \psi_+ + \theta_- \psi_- \qquad \begin{array}{l} \theta_+ = \theta \\ \theta_- = 1-\theta \end{array}.$$

Together with $\theta\delta \approx A\delta$ the weak equation $(\gamma^a \partial_a - m)\psi - [f, \not{p}\psi] \approx 0$ therefore entails

$$\not{p}\psi_+ - \not{p}\psi_- = A[\tilde{f}, \not{p}\psi_+] + (1-A)[\tilde{f}, \not{p}\psi_-]$$

or equivalently

$$(id - A[\tilde{f}, .])\not{p}\psi_+ = (id + (1-A)[\tilde{f}, .])\not{p}\psi_-. \tag{1}$$

The "ambiguity" (macroaspect of the product between θ and δ) A will be determined by a physical condition. The smooth Dirac-equation implies the existence of a conserved current J^a

$$J^a = [\bar{\psi}, \gamma^a \psi] \qquad \bar{\psi} := \psi^\dagger B \quad -B\gamma^a B^{-1} = \gamma^{a\dagger} \Leftrightarrow \bar{\gamma}^a = -\gamma^a$$

via

$$\begin{aligned} D_a[\bar{\psi}, \gamma^a \psi] &= [\overline{D_a \psi}, \gamma^a \psi] + [\bar{\psi}, \gamma^a D_a \psi] \\ &= -[\overline{\gamma^a D_a \psi}, \psi] + [\bar{\psi}, \gamma^a D_a \psi] = -m[\bar{\psi}, \psi] + m[\bar{\psi}, \psi] = 0, \end{aligned}$$

where in the first equality we took the reality of D_a with respect to YM-inner-product and in the second the imaginary character γ^a (anti-hermiticity with respect to the Dirac adjoint) into account. In the impulsive pp-wave context conservation involving non-linear operations no longer follows from the weak equation. Therefore requiring the weak conservation equation

$$\begin{aligned} D_a[\bar{\psi}, \gamma^a \psi] &= \partial_a[\bar{\psi}, \gamma^a \psi] + [f, [\bar{\psi}, \not{p}\psi]] \\ &= \partial_a[\bar{\psi}, \gamma^a \psi] + \delta[\tilde{f}, [\bar{\psi}, \not{p}\psi]] \approx 0 \end{aligned}$$

yields together with $\theta^2\delta \approx B\delta$

$$[\bar\psi_+(0),\not p\psi_+(0)]-[\bar\psi_-(0),\not p\psi_-(0)]+B[\tilde f,[\bar\psi_+(0),\not p\psi_+(0)]]+(A-B)\times$$
$$([\tilde f,[\bar\psi_+(0),\not p\psi_-(0)]]+[\tilde f,[\bar\psi_-(0),\not p\psi_+(0)]])+(1-2A+B)[\tilde f,[\bar\psi_-(0),\not p\psi_-(0)]]=0. \tag{2}$$

By taking the inner product of (2) with $\tilde f$ we find

$$tr(\tilde f[\bar\psi_+(0),\not p\psi_+(0)])-tr(\tilde f[\bar\psi_-(0),\not p\psi_-(0)])=0,$$

which yields together (1)

$$tr(\bar\psi_-(0)(id+(1-A)[\tilde f,.])^\dagger(id-A[\tilde f,.])^{-1\dagger}(id-A[\tilde f,.])^{-1}\times$$
$$(id+(1-A)[\tilde f,.])[\tilde f,\not p\psi_-(0)])=tr(\bar\psi_-(0)[\tilde f,\not p\psi_-(0)])$$

fixing $A=1/2$. This result, however, implies together with (2) $B=1/4$, thereby fixing both ambiguities. Therefore the junction condition reads in its final form

$$\not p\psi_+(0)=(id-\frac{1}{2}[\tilde f,.])^{-1}(id+\frac{1}{2}[\tilde f,.])\not p\psi_-(0),$$

where $U=(id-1/2[\tilde f,.])^{-1}(id+1/2[\tilde f,.])$ is unitary with respect to the inner product, being the Cayley-transform of the anti-hermitean operator $1/2[\tilde f,.]$ and no further restriction on $\psi_\pm$.

References

1. J.F. Colombeau, *Multiplication of Distributions: A tool in mathematics, numerical engineering and theoretical physics*, *Lecture Notes in Mathematics*, vol. 1532 (Springer, Berlin; New York, 1992)
2. H. Balasin, *Distributional Aspects of General Relativity: the Example of the Energy-Momentum Tensor of the Extended Kerr-geometry*, in *Nonlinear Theory of Generalized Functions*, *Research Notes in Mathematics*, vol. 401, ed. by M. Grosser, G. Hörmann, M. Kunzinger, M. Oberguggenberger (Chapman & Hall/CRC, London; Boca Raton, 1998), pp. 231–239
3. G. 't Hooft, *Graviton dominance in ultra-high-energy scattering*, Phys. Lett. B **198**, 61 (1987)
4. R. Penrose, *The geometry of impulsive gravitational waves*, in *General Relativity: Papers in Honour of J. L. Synge*, ed. by L. O'Raifeartaigh (Clarendon Press, Oxford, 1972), pp. 101–115
5. H. Balasin, *Geodesics for impulsive gravitational waves and the multiplication of distributions*, Class. Quantum Grav. **14**, 455 (1997)
6. M. Kunzinger, R. Steinbauer, *A rigorous solution concept for geodesic and geodesic deviation equations in impulsive gravitational waves*, J. Math. Phys. **40**, 1479 (1999)
7. N. Sanchez, H.J. de Vega, *Particle scattering at the Planck scale and the Aichelburg-Sexl geometry*, Nucl. Phys. **317B**, 731 (1989)